Theoretical and Mathematical Physics, **221**(3): 2207–2218 (2024)

ON THE EXISTENCE OF A NONEXTENDABLE SOLUTION OF THE CAUCHY PROBLEM FOR A (3 + 1)-DIMENSIONAL THERMAL–ELECTRICAL MODEL

M. V. Artemeva*† and M. O. Korpusov*†

A thermal–electrical $(3+1)$*-dimensional model of heating a semiconductor in an electric field is considered. For the corresponding Cauchy problem, the existence of a classical solution nonextendable in time is proved and an a priori estimate global in time is obtained.*

Keywords: nonlinear Sobolev-type equations, local solvability, nonlinear capacity, blow-up time estimates

DOI: 10.1134/S0040577924120146

1. Introduction

The modern radio information systems that provide solutions to space monitoring problems are characterized by a large number of densely located electronic units, continuous operation for a long time, and high reliability requirements. When a structurally complex radio information system operates in heat-stressed modes, the heat generation in electronic equipment increases sharply due to high current loads. The increased heat generation leads to overheating of the equipment and hence to a decrease in the reliability of the product [1] as well as an increase in the failure probability. These circumstances necessitate the need to study nonlinear thermal processes in a semiconductor and to construct and study the thermal–electrical model of a semiconductor.

This paper is a continuation of the studies started in [2]–[8]. In [7], a thermal–electrical semiconductor heating model was proposed, which reduced to considering the nonclassical third-order equation

$$\frac{\partial}{\partial t}\left(\phi_{xx}+\frac{\gamma}{2}|\phi_x|^2\right)+\frac{4\pi\sigma}{\varepsilon}\phi_{xx}=0. \tag{1}$$

For the problem on the closed interval $x\in[0,L]$ with the boundary conditions

$$\phi(0,t)=\mu_0(t),\qquad \phi_x(0,t)=\mu_1(t),\qquad \phi(x,0)=\phi_0(x) \tag{2}$$

we obtained the result on the existence of a classical solution nonextendable in time and also obtained sufficient conditions for the blow-up of solutions in finite time, which from a physical standpoint means the occurrence of electrical breakdown.

*Faculty of Physics, Lomonosov Moscow State University, Moscow, Russia,
e-mails: artemeva.mv14@physics.msu.ru (corresponding author), korpusov@gmail.com.

†Peoples' Friendship University of Russia, Moscow, Russia.

Translated from *Teoreticheskaya i Matematicheskaya Fizika*, Vol. 221, No. 3, pp. 702–715, December, 2024. Received April 26, 2024. Revised August 27, 2024. Accepted September 2, 2024.

In this paper, we consider the following Cauchy problem for a model (3 + 1)-dimensional equation in (1):

$$\frac{\partial}{\partial t}(\Delta_x u(x,t) + |D_x u(x,t)|^q) + \Delta_x u(x,t) = 0, \qquad u(x,0) = u_0(x), \tag{3}$$
$$\Delta_x := \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}.$$

For $q > 3/2$, we prove the existence of a nonextendable solution; for small initial data, we prove the existence of a global-in-time solution of the Cauchy problem and obtain an estimate for the decrease in time.

2. Cauchy Problem. The case $q \geqslant 2$

We consider the Cauchy problem

$$\begin{aligned} &\frac{\partial}{\partial t}(\Delta_x u + |D_x u|^q) + \Delta_x u = 0, \qquad q > 1, \quad (x,t) \in \mathbb{R}^3 \times [0,T], \\ &u(x,0) = u_0(x), \qquad x \in \mathbb{R}^3. \end{aligned} \tag{4}$$

We consider the class of radially symmetric solutions of Cauchy problem (4) and introduce a new function

$$w(r,t) := r^2 \frac{\partial u(r,t)}{\partial r}. \tag{5}$$

Then Cauchy problem (4) becomes

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\frac{\partial w(r,t)}{\partial r} + \frac{1}{r^{2(q-1)}}|w(r,t)|^q\right) + \frac{\partial w(r,t)}{\partial r} = 0, \\ &w(r,0) = r^2 \frac{\partial u_0(r)}{\partial r}. \end{aligned} \tag{6}$$

We thus arrive at the initial boundary value problem

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\frac{\partial w(r,t)}{\partial r} + \frac{|w|^q(r,t)}{r^{2(q-1)}}\right) + \frac{\partial w(r,t)}{\partial r} = 0, \qquad q > 1, \quad (r,t) \in [0,+\infty) \times [0,T], \\ &w(0,t) = 0, \qquad w(r,0) = w_0(r), \qquad (r,t) \in [0,+\infty) \times [0,T]. \end{aligned} \tag{7}$$

We consider the operator

$$Q_2(w)h(r) := \frac{\partial h(r)}{\partial r} + q\frac{|w(r)|^{q-2}w(r)}{r^{2(q-1)}}h(r) \tag{8}$$

and introduce Banach spaces required for the further study. We give the following definitions.

Definition 1. We say that $h \in C_b(r^{-\alpha}, 1 + r^{\gamma}; [0,+\infty))$ for $\alpha \geqslant 0$ and $\gamma \geqslant 0$ if $h \in C_b[0,+\infty)$ and the following norm is finite:

$$\|h\|_{\alpha,\gamma} := \sup_{r \in [0,+\infty)} \max\{1 + r^{\gamma}, r^{-\alpha}\}|h(r)|. \tag{9}$$

Remark 1. If $\gamma = 0$, then instead of $C_b(r^{-\alpha}, 1 + r^{\gamma}; [0,+\infty))$, we simply write $C_b(r^{-\alpha}; [0,+\infty))$, and instead of $\|\cdot\|_{\alpha,\gamma}$, we write $\|\cdot\|_{\alpha}$.

Definition 2. We say that $h \in C_b^{(1)}(r^{-\alpha},\, r^{-\beta},\, 1+r^{\gamma}\,;\,[0,+\infty))$ for $\alpha \geqslant 0$, $\beta \geqslant 0$, and $\gamma \geqslant 0$ if $h(r) \in C_b^{(1)}[0,+\infty)$ and the following norm is finite:

$$\|h\|_{\alpha,\beta,\gamma} := \sup_{r\in[0,+\infty)} \max\{1, r^{-\alpha}\}|h(r)| + \sup_{r\in[0,+\infty)} \max\{1+r^{\gamma}, r^{-\beta}\}\left|\frac{dh(r)}{dr}\right|. \tag{10}$$

The following lemmas hold.

Lemma 1. *The linear spaces*

$$C_b(r^{-\alpha}, 1+r^{\gamma}; [0,+\infty)), \qquad C_b^{(1)}(r^{-\alpha}, r^{-\beta}, 1+r^{\gamma}; [0,+\infty))$$

are Banach spaces with respect to respective norms (9) *and* (10).

Lemma 2. *For any function $w(r) \in C_b(r^{-\alpha}; [0,+\infty))$, if the inequalities*

$$\alpha \geqslant 2, \qquad q > 3/2 \tag{11}$$

are satisfied, then the operator $Q_2(w)$ acts as

$$\begin{gathered} Q_2(w)\colon C_b^{(1)}(r^{-\alpha}, r^{-\beta}, 1+r^{\gamma}; [0,+\infty)) \to C_b(r^{-\beta}, 1+r^{\gamma}; [0,+\infty)), \\ \beta = \alpha + (\alpha-2)(q-1), \qquad \gamma = 2(q-1), \end{gathered} \tag{12}$$

and the following relations hold:

$$\begin{gathered} Q_2^{-1}(w)\colon C_b(r^{-\beta}, 1+r^{\gamma}; [0,+\infty)) \to C_b^{(1)}(r^{-\alpha}, r^{-\beta}, 1+r^{\gamma}; [0,+\infty)), \\ h(r) := Q_2^{-1}(w)f(r) = \int_0^r \exp\biggl(-q\int_\rho^r \frac{|w(y)|^{q-2}w(y)}{y^{2(q-1)}}\,dy\biggr) f(\rho)\,d\rho. \end{gathered} \tag{13}$$

Proof. For $q > 3/2$ and $\alpha \geqslant 2$, we have the inequalities

$$\begin{gathered} \left|\frac{|w(r)|^{q-2}w(r)}{r^{2(q-1)}}\right| \leqslant \|w\|_\alpha^{q-1} r^{(\alpha-2)(q-1)} \leqslant \|w\|_\alpha^{q-1} \qquad \text{for all} \quad r \in [0,1], \\ \left|\frac{|w(r)|^{q-2}w(r)}{r^{2(q-1)}}\right| \leqslant \|w\|_\alpha^{q-1} \qquad \text{for all} \quad r \in [1,+\infty]. \end{gathered} \tag{14}$$

We note the estimate

$$\begin{aligned} \left|\int_\rho^r \frac{|w(y)|^{q-2}w(y)}{y^{2(q-1)}}\,dy\right| &\leqslant \int_0^{+\infty} \frac{|w(y)|^{q-1}}{y^{2(q-1)}}\,dy = \\ &= \int_0^1 \frac{|w(y)|^{q-1}}{y^{2(q-1)}}\,dy + \int_1^{+\infty} \frac{|w(y)|^{q-1}}{y^{2(q-1)}}\,dy \leqslant \\ &\leqslant \|w\|_\alpha^{q-1}\int_0^1 y^{(\alpha-2)(q-1)}\,dy + \|w\|_\alpha^{q-1}\int_1^{+\infty}\frac{1}{y^{2(q-1)}}\,dy \leqslant M_1(\alpha,q)\|w\|_\alpha^{q-1}. \end{aligned} \tag{15}$$

On the one hand, we note that from the explicit form of $h(r)$ in (22), it follows that $h(r) \in C^{(1)}[0,+\infty)$. On the other hand, for $r \in (0,1]$, we have

$$\begin{aligned} \frac{1}{r^{\beta+1}}|h(r)| &\leqslant \exp(qM_1(\alpha,q)\|w\|_\alpha^{q-1})\frac{\|f\|_{\beta,\gamma}}{r^{\beta+1}}\int_0^r \rho^\beta\,d\rho = \\ &= \exp(qM_1(\alpha,q)\|w\|_\alpha^{q-1})\frac{\|f\|_{\beta,\gamma}}{1+\beta}, \qquad \beta+1 > \alpha. \end{aligned} \tag{16}$$

For $r > 1$, we have

$$
\begin{aligned}
|h(r)| \leqslant \exp(M_1(\alpha,q)\|w\|_\alpha^{q-1})\|f\|_{\beta,\gamma}\int_0^1 \rho^\beta\, d\rho + \exp(M_1(\alpha,q)\|w\|_\alpha^{q-1})\|f\|_{\beta,\gamma}\times \\
\times\int_1^r \frac{d\rho}{1+\rho^\gamma} \leqslant M_2(q,\alpha)\exp(M_1(\alpha,q)\|w\|_\alpha^{q-1})\|f\|_{\beta,\gamma},
\end{aligned}
\tag{17}
$$

because $\gamma = 2(q-1) > 1$ for $q > 3/2$. Thus, from (16) and (17), we obtain $h(r) \in C_b(r^{-\alpha}; [0, +\infty))$. It is easy to prove the relation

$$
\frac{dh(r)}{dr} = f(r) - q\frac{|w(r)|^{q-2}w(r)}{r^{2(q-1)}}h(r) \in C_b(r^{-\beta}, 1 + r^\gamma; [0, +\infty)). \tag{18}
$$

The lemma is proved.

Lemma 3. *For any function $w(r,t) \in C([0,T]; C_b(r^{-\alpha}; [0,+\infty)))$, if the inequalities*

$$
\alpha \geqslant 2, \qquad q \geqslant 2 \tag{19}
$$

are satisfied, then the operator $Q_2(w)$ acts as

$$
\begin{aligned}
&Q_2(w)\colon C([0,T]; C_b^{(1)}(r^{-\alpha}, r^{-\beta}, 1 + r^\gamma; [0,+\infty))) \to C([0,T]; C_b(r^{-\beta}, 1 + r^\gamma; [0,+\infty))),\\
&\beta = \alpha + (\alpha - 2)(q-1), \qquad \gamma = 2(q-1),
\end{aligned}
\tag{20}
$$

and the following relations hold:

$$
Q_2^{-1}(w)\colon C([0,T]; C_b(r^{-\beta}, 1 + r^\gamma; [0,+\infty))) \to C([0,T]; C_b^{(1)}(r^{-\alpha}, r^{-\beta}, 1 + r^\gamma; [0,+\infty))), \tag{21}
$$

$$
h(r,t) := Q_2^{-1}(w)f(r,t)\int_0^r \exp\biggl(-q\int_\rho^r \frac{|w(y,t)|^{q-2}w(y,t)}{y^{2(q-1)}}\,dy\biggr) f(\rho,t)\,d\rho. \tag{22}
$$

Proof. Property (20) is obvious. We prove properties (21) and (22). The following relations hold:

$$
\begin{aligned}
h(r,t_1) - h(r,t_2) = \int_0^r \biggl[&\exp\biggl(-q\int_\rho^r \frac{|w(y,t_1)|^{q-2}w(y,t_1)}{y^{2(q-1)}}\,dy\biggr) - \\
&- \exp\biggl(-q\int_\rho^r \frac{|w(y,t_2)|^{q-2}w(y,t_2)}{y^{2(q-1)}}\,dy\biggr)\biggr] f(\rho,t_1)\,d\rho + \\
&+ \int_0^r \exp\biggl(-q\int_\rho^r \frac{|w(y,t_2)|^{q-2}w(y,t_2)}{y^{2(q-1)}}\,dy\biggr)\times \\
&\times [f(\rho,t_1) - f(\rho,t_2)]\,d\rho := h_1(r) + h_2(r),
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
&\exp\biggl(-q\int_\rho^r \frac{|w(y,t_1)|^{q-2}w(y,t_1)}{y^{2(q-1)}}\,dy\biggr) - \exp\biggl(-q\int_\rho^r \frac{|w(y,t_2)|^{q-2}w(y,t_2)}{y^{2(q-1)}}\,dy\biggr) = \\
&= \int_0^1 \frac{d}{ds}\exp\biggl(-q\int_\rho^r \frac{|w_s(y)|^{q-2}w_s(y)}{y^{2(q-1)}}\,dy\biggr)\,ds,\\
&w_s(y) := sw(y,t_1) + (1-s)w(y,t_2),
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
\frac{d}{ds}\exp\biggl(-q\int_\rho^r \frac{|w_s(y)|^{q-2}w_s(y)}{y^{2(q-1)}}\,dy\biggr) = -q(q-1)\exp\biggl(-q\int_\rho^r \frac{|w_s(y)|^{q-2}w_s(y)}{y^{2(q-1)}}\,dy\biggr)\times \\
\times\int_\rho^r \frac{|w_s(y)|^{q-2}}{y^{2(q-1)}}[w(y,t_1) - w(y,t_2)]\,dy.
\end{aligned}
\tag{25}
$$

First, we note the inequality

$$|w_s(y)| \leqslant s|w(y,t_1)| + (1-s)|w(y,t_2)| \leqslant \max\{|w(y,t_1)|, |w(y,t_2)|\}, \qquad s \in [0,1].$$

We then have the estimates

$$\begin{aligned} \left|\int_\rho^r \frac{|w_s(y)|^{q-2}}{y^{2(q-1)}}[w(y,t_1) - w(y,t_2)]\,dy\right| &\leqslant \int_0^1 \frac{|w_s(y)|^{q-2}}{y^{2(q-1)}}|w(y,t_1) - w(y,t_2)|\,dy + \\ &+ \int_1^{+\infty} \frac{|w_s(y)|^{q-2}}{y^{2(q-1)}}|w(y,t_1) - w(y,t_2)|\,dy \leqslant \max\{\|w(t_1)\|_\alpha^{q-2}, \|w(t_2)\|_\alpha^{q-2}\} \times \\ &\times \left[\int_0^1 y^{(\alpha-2)(q-1)}\,dy + \int_1^{+\infty} \frac{dy}{y^{2(q-1)}}\right]\|w(t_1) - w(t_2)\|_\alpha \leqslant \\ &\leqslant M_3(\alpha,q)\max\{\|w(t_1)\|_\alpha^{q-2}, \|w(t_2)\|_\alpha^{q-2}\}\|w(t_1) - w(t_2)\|_\alpha. \end{aligned} \tag{26}$$

Thus, from (23) with (26) taken into account, we obtain

$$\begin{aligned} \|h_1\|_\alpha \leqslant M_4(\alpha,q)\exp(M_1(\alpha,q)\max\{\|w(t_1)\|_\alpha^{q-1}, \|w(t_2)\|_\alpha^{q-1}\}) \times \\ \times \max\{\|w(t_1)\|_\alpha^{q-2}, \|w(t_2)\|_\alpha^{q-2}\}\|w(t_1) - w(t_2)\|_\alpha \|f(t_1)\|_{\beta,\gamma}. \end{aligned} \tag{27}$$

Similarly, we obtain the estimate

$$\|h_2\|_\alpha \leqslant M_4(\alpha,q)\exp(M_1(\alpha,q)\|w(t_2)\|_\alpha^{q-1})\|f(t_1) - f(t_2)\|_{\beta,\gamma}. \tag{28}$$

Therefore, Eqs. (27) and (28) imply the estimate

$$\begin{aligned} \|h(t_1) - h(t_2)\|_\alpha \leqslant M_4(\alpha,q)[&\exp(M_1(\alpha,q)\max\{\|w(t_1)\|_\alpha^{q-1}, \|w(t_2)\|_\alpha^{q-1}\}) \times \\ &\times \max\{\|w(t_1)\|_\alpha^{q-2}, \|w(t_2)\|_\alpha^{q-2}\}\|w(t_1) - w(t_2)\|_\alpha \|f(t_1)\|_{\beta,\gamma} + \\ &+ \exp(M_1(\alpha,q)\|w(t_2)\|_\alpha^{q-1})\|f(t_1) - f(t_2)\|_{\beta,\gamma}], \end{aligned} \tag{29}$$

whence it follows that

$$\|h(t_1) - h(t_2)\|_\alpha \to +0 \qquad \text{as} \quad |t_1 - t_2| \to +0. \tag{30}$$

We now note the relation

$$\frac{dh(r,t)}{dr} = f(r,t) - q\frac{|w(r,t)|^{q-2}w(r,t)}{r^{2(q-1)}}h(r,t), \tag{31}$$

from which, similarly to (30), we obtain the estimate

$$\begin{aligned} \left\|\frac{dh(r,t_1)}{dr} - \frac{dh(r,t_2)}{dr}\right\|_{\beta,\gamma} &\leqslant \|f(r,t_1) - f(r,t_2)\|_{\beta,\gamma} + q\|w(t_1)\|_\alpha^{q-1}\|h(t_1) - h(t_2)\|_\alpha + \\ &+ q(q-1)\max\{\|w(t_1)\|_\alpha^{q-2}, \|w(t_2)\|_\alpha^{q-2}\}\|h(t_2)\|_\alpha \|w(t_1) - w(t_2)\|_\alpha, \end{aligned} \tag{32}$$

which implies

$$\left\|\frac{dh(r,t_1)}{dr} - \frac{dh(r,t_2)}{dr}\right\|_{\beta,\gamma} \to +0 \qquad \text{as} \quad |t_1 - t_2| \to +0. \tag{33}$$

Thus, from (30) and (33), we obtain

$$\|h(t_1) - h(t_2)\|_{\alpha,\beta,\gamma} \to +0 \qquad \text{as} \quad |t_1 - t_2| \to +0. \tag{34}$$

Therefore, $h(r,t) \in C([0,T]; C_b^{(1)}(r^{-\alpha}, r^{-\beta}, 1 + r^\gamma; [0,+\infty)))$. The lemma is proved.

We now define the classical solution of initial boundary value problem (7).

Definition 3. A function $w(r,t) \in C^{(1)}([0,T]; C_b^{(1)}(r^{-\alpha}, r^{-\beta}, 1+r^{\gamma}; [0,+\infty)))$ with

$$\alpha \geqslant 2, \quad q > 1, \quad \beta = \alpha + (\alpha-2)(q-1), \quad \gamma = 2(q-1)$$

is called a classical solution of problem (7) if the function pointwise satisfies the above relations for all $(r,t) \in [0,+\infty) \times [0,T]$, and the derivatives at the boundary points are understood as one-sided limits.

Let $w(r,t)$ be a classical solution of problem (7) in the sense of Definition 1, and let $q \geqslant 2$. Then the following equivalent relations hold for all $(r,t) \in [0,+\infty) \times [0,T]$:

$$Q_2(w)\frac{\partial w}{\partial t} + Q_2(w)w = f(r,t), \qquad w(r,0) = w_0(r), \qquad f(r,t) := q\frac{|w(r,t)|^q}{r^{2(q-1)}}, \tag{35}$$

$$\frac{\partial w}{\partial t} + w = Q_2^{-1}(w)f(r,t), \qquad w(r,0) = w_0(r), \tag{36}$$

$$w(t) = w_0 e^{-t} + \int_0^t e^{-(t-\tau)} Q_2^{-1}(w(\tau)) f(\tau)\, d\tau, \qquad f(r,t) := q\frac{|w(r,t)|^q}{r^{2(q-1)}}. \tag{37}$$

We can rewrite the last integral equation in the form

$$w(t) = Q(w)(t), \tag{38}$$

$$Q(w)(t) := w_0 e^{-t} + \int_0^t e^{-(t-\tau)} Q_2^{-1}(w(\tau)) f(\tau)\, d\tau, \qquad f(r,t) := q\frac{|w(r,t)|^q}{r^{2(q-1)}}. \tag{39}$$

The following lemma holds.

Lemma 4. *For $q \geqslant 2$ and any function $w_0(r) \in C_b^{(1)}(r^{-\alpha}, r^{-\beta}, 1+r^{\gamma}; [0,+\infty))$, the operator defined in* (39) *acts as*

$$Q\colon C([0,T]; C_b(r^{-\alpha}; [0,+\infty))) \to C^{(1)}([0,T]; C_b^{(1)}(r^{-\alpha}, r^{-\beta}, 1+r^{\gamma}; [0,+\infty))). \tag{40}$$

Proof. On one hand, we note that

$$f(r,t) \in C([0,T]; C_b(r^{-\beta}, 1+r^{\gamma}; [0,+\infty)) \tag{41}$$

for any function

$$w(r,t) \in C([0,T]; C_b(r^{-\alpha}; [0,+\infty))). \tag{42}$$

On the other hand, the operator

$$S\phi(t) := \int_0^t e^{-(t-\tau)} \phi(\tau)\, d\tau \tag{43}$$

act as

$$S\colon C([0,T]; C_b^{(1)}(r^{-\alpha}, r^{-\beta}, 1+r^{\gamma}; [0,+\infty))) \to C^{(1)}([0,T]; C_b^{(1)}(r^{-\alpha}, r^{-\beta}, 1+r^{\gamma}; [0,+\infty))). \tag{44}$$

It remains to note that

$$Q(w) = w_0(r)e^{-t} + S(Q_2^{-1}(w) f(r,t)), \qquad f(r,t) := q\frac{|w(r,t)|^q}{r^{2(q-1)}}. \tag{45}$$

Remark 2. In particular, we have

$$Q\colon C([0,T];C_b(r^{-\alpha};[0,+\infty)))\to C([0,T];C_b(r^{-\alpha};[0,+\infty))). \tag{46}$$

We consider the closed convex and bounded set

$$\begin{aligned}&B_R:=\{w(t)\in C([0,T];C_b(r^{-\alpha};[0,+\infty))):\ \|w\|\leqslant R\},\\&\|w\|:=\sup_{t\in[0,T]}\|w(t)\|_\alpha.\end{aligned} \tag{47}$$

The following lemma holds.

Lemma 5. *For any function $w_0(r)\in C_b(r^{-\alpha};[0,+\infty))$ and for $q>1$, there exists a sufficiently large $R>0$ and a sufficiently small $T>0$ such that*

$$Q\colon B_R\to B_R. \tag{48}$$

Proof. Let $R>0$ be large enough that

$$\|w_0\|=\|w_0\|_\alpha\leqslant\frac{R}{2}. \tag{49}$$

We fix such an $R>0$. The following estimate holds:

$$\|Q_2^{-1}(w(\tau))f(\tau)\|_\alpha\leqslant M_4(\alpha,q)e^{M_1(\alpha,q)\|w\|^{q-1}}\|w\|^q. \tag{50}$$

It follows from (39) with (49) and (50) taken into account that the estimate

$$\begin{aligned}\|Q(w)\|&\leqslant\|w_0\|+TM_4(\alpha,q)e^{(M_1(\alpha,q)\|w\|^{q-1}}\|w\|^q\leqslant\\&\leqslant\frac{R}{2}+TM_4(\alpha,q)e^{M_1(\alpha,q)R^{q-1}}R^q\leqslant R,\end{aligned} \tag{51}$$

holds if $T>0$ is small enough that

$$TM_4(\alpha,q)e^{M_1(\alpha,q)R^{q-1}}R^{q-1}\leqslant\frac{1}{2}. \tag{52}$$

Taking Lemma 4 into account, we obtain the sought assertion. The lemma is proved.

The following lemma holds.

Lemma 6. *For a sufficiently small $T>0$ and for $q\geqslant 2$, Q is a contraction operator on B_R:*

$$\|Q(w_1)-Q(w_2)\|\leqslant\frac{1}{2}\|w_1-w_2\| \tag{53}$$

for any $w_1,w_2\in B_R$.

Proof. Let $w_1(t),w_2(t)\in B_R$. We introduce the following functions for $k=1,2$:

$$g_k(r,t):=(Q_2^{-1}(w_k(r,t))f_k(r,t))(r,t),\qquad f_k(r,t):=q\frac{|w_k(r,t)|^q}{r^{2(q-1)}}. \tag{54}$$

As in the derivation of estimate (29), we obtain the inequality

$$\|g_1(r,t) - g_2(r,t)\| \leqslant M_4(\alpha,q)[e^{M_1(\alpha,q)R^{q-1}} R^{q-2}\|w_1 - w_2\| \sup_{t\in[0,T]} \|f_1\|_{\beta,\gamma} + \\ + e^{M_1(\alpha,q)R^{q-1}} \sup_{t\in[0,T]} \|f_1 - f_2\|_{\beta,\gamma}], \tag{55}$$

and the following estimate holds:

$$\sup_{t\in[0,T]} \|f_1(t) - f_2(t)\|_{\beta,\gamma} \leqslant M_5(\alpha,q)R^{q-1}\|w_1 - w_2\|. \tag{56}$$

From estimates (55) and (56), we obtain

$$\|g_1(r,t) - g_2(r,t)\| \leqslant M_6(\alpha,q)e^{M_1(\alpha,q)R^{q-1}} R^{q-1}[R^q + 1]\|w_1 - w_2\|. \tag{57}$$

From (39) with (56) and (57) taken into account, we obtain the estimate

$$\|Q(w_1) - Q(w_2)\| \leqslant TM_6(\alpha,q)e^{M_1(\alpha,q)R^{q-1}} R^{q-1}[R^q + 1]\|w_1 - w_2\| \leqslant \frac{1}{2}\|w_1 - w_2\| \tag{58}$$

if $T > 0$ is small enough that

$$TM_6(\alpha,q)e^{M_1(\alpha,q)R^{q-1}} R^{q-1}[R^q + 1] \leqslant \frac{1}{2}.$$

The lemma is proved.

Taking Lemmas 5 and 6 and the contraction mapping principle into account, we conclude that for any function $w_0(r) \in C_b(r^{-\alpha};[0,+\infty))$, if inequalities (19) are satisfied for sufficiently small $T > 0$, there exists a unique solution of integral equation (38) in the class $C([0,T];C_b(r^{-\alpha};[0,+\infty)))$. Using standard algorithms for continuing solutions of integral equation (38) in time (see [9]), we obtain the following result.

Theorem 1. *If $q \geqslant 2$ and the inequality $\alpha > 2$ is satisfied, then for any function $w_0(r) \in C_b(r^{-\alpha};[0,+\infty))$, there is a maximal $T_0 = T_0(w_0) > 0$ such that, for any $T \in (0,T_0)$, there exists a unique solution $w(r,t)$ of class $C([0,T];C_b(r^{-\alpha};[0,+\infty)))$ of integral equation* (38)*, and either $T_0 = +\infty$ or $T_0 < +\infty$; in the latter case,*

$$\lim_{T\uparrow T_0} \|w(t)\|_\alpha = +\infty. \tag{59}$$

Taking Lemma 4 into account and using Eq. (38), we formulate the main theorem in this paper.

Theorem 2. *If $q \geqslant 2$ and inequality $\alpha > 2$ is satisfied, then for any function $w_0(r) \in C_b(r^{-\alpha};[0,+\infty)) \cap C^{(1)}[0,+\infty)$, there is a maximal $T_0 = T_0(w_0) > 0$ such that, for any $T \in (0,T_0)$, there exists a unique classical solution of problem* (7) *in the sense of Definition* 3*, and either $T_0 = +\infty$ or $T_0 < +\infty$; in the latter case, the limit property* (59) *holds.*

We now consider the following integral inequality for $t \in [0,T]$:

$$z(t) \leqslant z(0)e^{-t} + \int_0^t e^{-(t-\tau)}a_1e^{a_2z^{q-1}(\tau)}z^q(\tau)\,d\tau, \qquad q > 1, \quad z(t) \geqslant 0. \tag{60}$$

We assume that the number $d > 0$ is such that the inequalities

$$z(0) < d, \qquad a_1e^{a_2d^{q-1}}d^{q-1} \leqslant 1 \tag{61}$$

hold. We also assume that there is $t = t_0 > 0$ such that

$$z(t) < d, \qquad z(t_0) = d \qquad \text{for all} \quad t \in [0, t_0). \tag{62}$$

Then we have

$$a_1 e^{a_2 z^{q-1}(t)} z^q(t) \leqslant a_1 e^{a_2 d^{q-1}} d^{q-1} d \leqslant d \qquad \text{for} \quad t \in [0, t_0]. \tag{63}$$

Therefore, from (60) with (61) and (63) taken into account, we obtain the inequality

$$z(t_0) \leqslant z(0)e^{-t_0} + \int_0^{t_0} e^{-(t_0-\tau)} a_1 e^{a_2 z^{q-1}(\tau)} z^q(\tau)\, d\tau <$$
$$< de^{-t_0} + d(1 - e^{-t_0}) = d \quad \Rightarrow \quad z(t_0) < d. \tag{64}$$

This contradicts assumption (62). Therefore, if inequality (61) holds, we have

$$z(t) < d \quad \text{for all} \quad t \in [0, T]. \tag{65}$$

The following theorem holds.

Theorem 3. *If in addition to the conditions in Theorem* 2, *we have*

$$\|w_0\|_\alpha < d \quad \textit{for} \quad M_4(\alpha, q) e^{M_1(\alpha,q) d^{q-1}} d^{q-1} \leqslant 1, \tag{66}$$

where M_1 and M_4 are constants that appear in the proof of Lemma 2, *then the classical solution of Cauchy problem* (7) *in the sense of Definition* 3 *exists globally in time, and $\|w(t)\|_\alpha < d$ for all $t \in [0, +\infty)$. If*

$$w_0(r) \geqslant 0, \qquad M_4(\alpha, q)\|w_0\|_\alpha^{q-1} < 1, \tag{67}$$

then the solution of the Cauchy problem exists globally in time and the following inequality holds:

$$\|w\|_\alpha(t) \leqslant \frac{\|w_0\|_\alpha e^{-t}}{[1 - M_4(\alpha, q)\|w_0\|_\alpha^{q-1}(1 - e^{-(q-1)t})]^{1/(q-1)}}. \tag{68}$$

Proof. As in the proof of Lemma 2, we obtain the inequality

$$\|w\|_\alpha(t) \leqslant \|w_0\|_\alpha e^{-t} + \int_0^t e^{-(t-\tau)} M_4(\alpha, q) e^{M_1(\alpha,q)\|w\|_\alpha^{q-1}(\tau)} \|w\|_\alpha^q(\tau)\, d\tau. \tag{69}$$

It remains to use the arguments in (60)–(65).

If $w_0(r) \geqslant 0$ in addition to the other conditions, then integral equation (37) implies the inequality $w(r, t) \geqslant 0$ for all $(r, t) \in [0, +\infty) \times [0, T_0)$. With this inequality taken into account, we use integral equation (38) to obtain the estimate

$$\|w\|_\alpha(t) \leqslant \|w_0\|_\alpha e^{-t} + M_4(\alpha, q) \int_0^t e^{-(t-\tau)} \|w\|_\alpha^q(\tau)\, d\tau. \tag{70}$$

We introduce the function

$$z(t) := \|w\|_\alpha(t) e^t. \tag{71}$$

From (70), we then derive the inequality

$$z(t) \leqslant z(0) + M_4 e^{-t} \int_0^t e^{-(q-1)\tau} z^q(\tau)\, d\tau \leqslant z(0) + M_4 \int_0^t e^{-(q-1)\tau} z^q(\tau)\, d\tau, \tag{72}$$

whence, using the Bihari inequality (see [10]), we obtain the inequality

$$z(t) \leqslant \frac{z(0)}{[1 - M_4(z(0))^{q-1}(1 - e^{-(q-1)t})]^{1/(q-1)}}, \tag{73}$$

from which, using the condition $M_4 z^{q-1}(0) < 1$, we deduce the remaining assertions of the theorem. The theorem is proved.

3. Cauchy Problem. The case $3/2 < q < 2$

In the case $3/2 < q < 2$, Lemma 2 remains true. Lemma 3 takes the following form.

Lemma 7. *For any function $w(r,t) \in C([0,T]; C_b(r^{-\alpha}; [0,+\infty)))$ if the inequalities $\alpha \geqslant 2$, $3/2 < q < 2$, and*

$$\begin{aligned} &w(r,t) \geqslant w_0(r) e^{-t}, \qquad w_0(r) \in C_b(r^{-\alpha}; [0,+\infty)), \\ &w_0(r) \geqslant a_0 \min\{1, r^\alpha\}, \qquad a_0 > 0, \end{aligned} \tag{74}$$

are satisfied, then the operator $Q_2(w)$ acts as

$$\begin{aligned} &Q_2(w)\colon C([0,T]; C_b^{(1)}(r^{-\alpha}, r^{-\beta}, 1 + r^\gamma; [0,+\infty))) \to C([0,T]; C_b(r^{-\beta}, 1 + r^\gamma; [0,+\infty))), \\ &\beta = \alpha + (\alpha - 2)(q-1), \qquad \gamma = 2(q-1), \end{aligned} \tag{75}$$

and the following relations hold:

$$\begin{aligned} &Q_2^{-1}(w)\colon C([0,T]; C_b(r^{-\beta}, 1 + r^\gamma; [0,+\infty))) \to C([0,T]; C_b^{(1)}(r^{-\alpha}, r^{-\beta}, 1 + r^\gamma; [0,+\infty))), \\ &h(r,t) := Q_2^{-1}(w) f(r,t) = \int_0^r \exp\biggl(-q \int_\rho^r \frac{|w(y,t)|^{q-2} w(y,t)}{y^{2(q-1)}}\, dy\biggr) f(\rho,t)\, d\rho. \end{aligned} \tag{76}$$

Proof. The proof of this lemma repeats the proof of Lemma 2. We must consider estimate (26) separately, where the use of condition $q \geqslant 2$ was essential. In the case $3/2 < q < 2$, this estimate becomes

$$\begin{aligned} &w_s(y) = s w(y,t_1) + (1-s) w(y,t_2) \geqslant s w_0(y) e^{-t_1} + (1-s) w_0(y) e^{-t_2} = \\ &\qquad = w_0(y) \min\{e^{-t_1}, e^{-t_2}\} \geqslant a_0 \min\{1, y^\alpha\} \min\{e^{-t_1}, e^{-t_2}\} \geqslant 0, \quad s \in [0,1],\ y \geqslant 0, \\ &|w_s(y)|^{q-2} |w(y,t_1) - w(y,t_2)| \leqslant \\ &\leqslant a_0^{q-2} y^{\alpha(q-1)} \max\{e^{(2-q)t_1}, e^{(2-q)t_2}\} \|w(t_1) - w(t_2)\|_\alpha, \qquad y \in [0,1], \\ &|w_s(y)|^{q-2} |w(y,t_1) - w(y,t_2)| \leqslant \\ &\leqslant a_0^{q-2} \max\{e^{(2-q)t_1}, e^{(2-q)t_2}\} \|w(t_1) - w(t_2)\|_\alpha, \qquad y \geqslant 1, \\ &\biggl|\int_\rho^r \frac{|w_s(y)|^{q-2}}{y^{2(q-1)}} [w(y,t_1) - w(y,t_2)]\, dy\biggr| \leqslant \int_0^1 \frac{|w_s(y)|^{q-2}}{y^{2(q-1)}} |w(y,t_1) - w(y,t_2)|\, dy + \\ &\qquad + \int_1^{+\infty} \frac{|w_s(y)|^{q-2}}{y^{2(q-1)}} |w(y,t_1) - w(y,t_2)|\, dy \leqslant \\ &\leqslant a_0^{q-2} \max\{e^{(2-q)t_1}, e^{(2-q)t_2}\} \biggl[\int_0^1 y^{(\alpha-2)(q-1)}\, dy + \int_1^{+\infty} \frac{dy}{y^{2(q-1)}}\biggr] \times \\ &\qquad \times \|w(t_1) - w(t_2)\|_\alpha \leqslant M_3(\alpha, q, a_0) \max\{e^{(2-q)t_1}, e^{(2-q)t_2}\} \|w(t_1) - w(t_2)\|_\alpha. \end{aligned}$$

Similarly, we must change estimate (32). Thus, the lemma is proved.

We now consider the following complete metric space with the metric generated by the norm

$$\begin{aligned} &D_R := \{w(t) \in C([0,T]; C_b(r^{-\alpha}; [0,+\infty)))\colon \|w\| \leqslant R,\ w(t) \geqslant w_0 e^{-t}\}, \\ &\|w\| := \sup_{t \in [0,T]} \|w(t)\|_\alpha, \qquad w_0 \in C_b(r^{-\alpha}; [0,+\infty)). \end{aligned} \tag{77}$$

Lemmas 4 and 5 with B_R replaced by D_R remain valid without changes. The following lemma holds.

Lemma 8. *If conditions (74) are satisfied for a sufficiently small $T > 0$ and for $q \in (3/2, 2)$, then Q is a contraction operator on D_R:*

$$\|Q(w_1) - Q(w_2)\| \leqslant \frac{1}{2}\|w_1 - w_2\| \tag{78}$$

for any $w_1, w_2 \in D_R$.

Proof. Let $w_1(t), w_2(t) \in D_R$. We introduce the following functions for $k = 1, 2$:

$$\begin{gathered}
g_k(r,t) := \Bigl(Q_2^{-1}(w_k(r,t)) f_k(r,t)\Bigr)(r,t), \qquad f_k(r,t) := q\frac{|w_k(r,t)|^q}{r^{2(q-1)}},\\
g_1(r,t) - g_2(r,t) = \int_0^r \biggl[\exp\biggl(-q\int_\rho^r \frac{|w_1(y,t)|^{q-2}w_1(y,t)}{y^{2(q-1)}}\,dy\biggr) - \\
- \exp\biggl(-q\int_\rho^r \frac{|w_2(y,t)|^{q-2}w_2(y,t)}{y^{2(q-1)}}\,dy\biggr)\biggr] f_1(\rho,t)\,d\rho + \\
+ \int_0^r \exp\biggl(-q\int_\rho^r \frac{|w_2(y,t)|^{q-2}w_2(y,t)}{y^{2(q-1)}}\,dy\biggr) \times \\
\times [f_1(\rho,t) - f_2(\rho,t)]\,d\rho := h_1(r,t) + h_2(r,t),\\
\exp\biggl(-q\int_\rho^r \frac{|w_1(y,t)|^{q-2}w_1(y,t)}{y^{2(q-1)}}\,dy\biggr) - \exp\biggl(-q\int_\rho^r \frac{|w_2(y,t)|^{q-2}w_2(y,t)}{y^{2(q-1)}}\,dy\biggr) = \\
= \int_0^1 \frac{d}{ds}\exp\biggl(-q\int_\rho^r \frac{|w_s(y,t)|^{q-2}w_s(y,t)}{y^{2(q-1)}}\,dy\biggr)\,ds,\\
w_s(y,t) := sw_1(y,t) + (1-s)w_2(y,t),\\
\frac{d}{ds}\exp\biggl(-q\int_\rho^r \frac{|w_s(y,t)|^{q-2}w_s(y,t)}{y^{2(q-1)}}\,dy\biggr) = -q(q-1)\exp\biggl(-q\int_\rho^r \frac{|w_s(y,t)|^{q-2}w_s(y,t)}{y^{2(q-1)}}\,dy\biggr) \times \\
\times \int_\rho^r \frac{|w_s(y,t)|^{q-2}}{y^{2(q-1)}}[w_1(y,t) - w_2(y,t)]\,dy.
\end{gathered} \tag{79}$$

It now remains to use the estimates

$$\begin{gathered}
w_s(y,t) = sw_1(y,t) + (1-s)w_2(y,t) \geqslant sw_0(y)e^{-t} + (1-s)w_0(y)e^{-t} = \\
= w_0(y)e^{-t} \geqslant a_0\min\{1, y^\alpha\}e^{-t} \geqslant 0, \qquad s \in [0,1], \quad y \geqslant 0,\\
|w_s(y,t)|^{q-2}|w_1(y,t) - w_2(y,t)| \leqslant a_0^{q-2}y^{\alpha(q-1)}e^{(2-q)t}\|w_1(t) - w_2(t)\|_\alpha, \quad y \in [0,1],\\
|w_s(y,t)|^{q-2}|w(y,t) - w(y,t)| \leqslant a_0^{q-2}e^{(2-q)t}\|w_1(t) - w_2(t)\|_\alpha, \quad y \geqslant 1,\\
\biggl|\int_\rho^r \frac{|w_s(y,t)|^{q-2}}{y^{2(q-1)}}[w_1(y,t) - w_2(y,t)]\,dy\biggr| \leqslant \int_0^1 \frac{|w_s(y,t)|^{q-2}}{y^{2(q-1)}}|w_1(y,t) - w_2(y,t)|\,dy + \\
+ \int_1^{+\infty} \frac{|w_s(y,t)|^{q-2}}{y^{2(q-1)}}|w_1(y,t) - w_2(y,t)|\,dy \leqslant a_0^{q-2}e^{(2-q)t}\biggl[\int_0^1 y^{(\alpha-2)(q-1)}\,dy + \int_1^{+\infty}\frac{dy}{y^{2(q-1)}}\biggr] \times \\
\times \|w_1(t) - w_2(t)\|_\alpha \leqslant M_3(\alpha, q, a_0)e^{(2-q)t}\|w_1(t) - w_2(t)\|_\alpha.
\end{gathered}$$

The further reasoning is the same as in the proof of Lemma 6. The lemma is proved.

Further, as in the preceding section, we conclude that Theorems 1 and 2 hold under the condition $q \in (3/2, 2)$ and conditions (74). Finally, under these conditions, Theorem 3 is also true.

Funding. This work was carried out with the financial support of the Russian Science Foundation (project No. 23-11-00056).

Conflict of interest. The authors of this work declare that they have no conflicts of interest.

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