Once More on Typicality and Atypicality of Power-Law Asymptotic Behavior of Solutions to Emden–Fowler Type Differential Equations

I. Astashova

Lomonosov Moscow State University,
Plekhanov Russian University of Economics
ast.diffiety@gmail.com

Consider the equation

$$y^{(n)} = p(x, y, y', \dots, y^{(n-1)})|y|^k \operatorname{sign} y, \tag{1}$$

where $n \geq 2$, $k > 1$, and p is a positive continuous function that is Lipschitz-continuous in its last n variables. Also consider a special case of (1), namely

$$y^{(n)} = p_0\, |y|^k \operatorname{sign}\, y \tag{2}$$

with $p_0 > 0$.

Immediate calculations show that equation (2) has positive solutions with exact power-law behavior, namely

$$y(x) = C(x^* - x)^{-\alpha} \tag{3}$$

defined on $(-\infty,\, x^*)$ with

$$\alpha = \frac{n}{k-1}, \quad C = \left(\frac{\alpha(\alpha+1)\dots(\alpha+n-1)}{p_0}\right)^{\frac{1}{k-1}}, \tag{4}$$

and arbitrary $x^* \in \mathbb{R}$.

We discuss the problem posed by I. Kiguradze (see [1], problem 16.4) on asymptotic behavior of all positive non-extensible (so-called “blow-up”) solutions to equations (2) and (1).

For $n = 2$ (see [1]), $n = 3, 4$ (see [2],**5.1**–[4]), it appears that if $p(x, y_1, y_2, \dots, y_{n-1})$ tends to p_0 as $x \to x^* - 0, y_0 \to \infty, \cdots, y_{n-1} \to \infty$, then all such solutions to equation (1) (and equation (2)) have the following power-law asymptotic behavior:

$$y(x) = C(x^* - x)^{-\alpha}(1 + o(1)), \quad x \to x^* - 0, \tag{5}$$

with α and C defined by (4).

For equation (1) with any n and some additional assumptions on the function p, the existence of solutions with power-law asymptotic behavior is proved, for $5 \leq n \leq 11$, the existence of an $(n-1)$-parametric family of such solutions is obtained.(see [2](**5.1**)) .

It is also proved that for weakly super-linear equations (2) (see [5]) and (1) (see [6]) Kiguradze’s conjecture on the power-law asymptotic behavior of all blow-up solutions is true.

Theorem 1 ([6]). *Suppose that $p \in C(\mathbb{R}^{n+1}) \bigcap Lip_{y_0,\dots,y_{n-1}}(\mathbb{R}^n)$ and $p \to p_0 > 0$ as $x \to x^*, y_0 \to \infty, \dots, y_{n-1} \to \infty$. Then for any integer $n > 4$ there exists $K_n > 1$ such that for any real $k \in (1, K_n)$, any solution to equation (1) tending to $+\infty$ as $x \to x^* - 0$ has the power-law asymptotic behavior (5).*

In the case $n \geq 12$, even if we deal with equation (2), another type of asymptotic behavior of singular solutions appears (see [6]–[9]).

Theorem 2 ([9]). *For any $n \geq 12$, there exists $k_n > 1$ such that equation (2) has a solution $y(x)$ with*

$$y^{(j)}(x) = p_0^{-\frac{1}{k-1}} (x^* - x)^{-\alpha - j} \, h_j(\, \log(x^* - x)\,), \qquad j = 0, 1, \ldots, n-1, \tag{6}$$

where all h_j are periodic positive non-constant functions on $\mathbb{R}$.

If we have stronger nonlinearity, then the power-law asymptotic behavior becomes atypical. The following theorem generalizes the results of [10].

Theorem 3. *If $12 \leq n \leq 100000$, then there exists $k_n > 1$ such that at any point $x_0 \in \mathbb{R}$ the set of initial data of asymptotically power-law solutions to equation (2) has zero Lebesgue measure whenever $k > k_n$.*

In order to study the blow-up solutions to equation (2) having the vertical asymptote $x = x^*$, we use the substitutions

$$x^* - x = e^{-t}, \qquad y = (C + v)\, e^{\alpha t} \tag{7}$$

with C defined by (4) to transform equation (2) with $p_0 = 1$ to another one, which can be reduced to the first-order system

$$\frac{dV}{dt} = A_\alpha V + F_\alpha(V), \tag{8}$$

where A_α is a constant $n \times n$ matrix with eigenvalues satisfying the equation

$$\prod_{j=0}^{n-1} (\lambda + \alpha + j) = \prod_{j=0}^{n-1} (1 + \alpha + j), \tag{9}$$

and F_α is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ satisfying $||F_\alpha(V)|| = O\left(||V||^2\right)$ and $||F'_{\alpha,V}(V)|| = O\left(||V||\right)$ as $V \to 0$. In order to study equation (1), the same substitution as (7) of variables is used, and a more complicated system then (8) with an additional term $G(t, V)$ appears (see [2]).

The proof of Theorem 3 is based on the following statement.

Lemma 1. *If there is no purely imaginary root to equation (9), but there exists at least one root not equal to 1 and having positive real part, then for any $x_0 \in \mathbb{R}$, the set of initial data of asymptotically power-law solutions to equation (2) has zero Lebesgue measure whenever $k > k_n$.*

Remark 1. The occurrence of the order 12 for equation (2) in Theorems 2 and 3 is explained by the fact that all roots but one ($\lambda = 1$) to equation (9) with $n < 12$ have negative real parts, which implies the existence of an $(n-1)$-parametric family of solutions with power-law asymptotic behavior (5) of solutions to equation (1). Equation (9) with $n = 12$ and some α has a pair of complex-conjugate purely imaginary roots, which implies the appearance of a solution of the form (6) to equation (2). The order 100000 appearing in Theorem 3 is not final. It is possible to continue the calculations and obtain the same result for equations of order higher than 100000. The previous result (see [10]) was obtained for $12 \leq n \leq 203$.

Open problems.

- Is there any blow-up solution to equation (2) with asymptotic behavior other than (5) and (6)?

- Is there any blow-up solution with non-power-law asymptotic behavior to equation (2) with strong power-law nonlinearity when $5 \le n \le 11$?
- Is it possible to find exactly a constant $K_n^* > 1$ such that for any $k \in (1, K_n^*)$ all blow-up solutions to (2) have power-law asymptotic behavior (5), while other blow-up solutions appear whenever $k > K_n^*$?

The research was partially supported by Russian Science Foundation (scientific project 20-11-20272).

References

[1] Kiguradze, I. T., Chanturia, T. A.: Asymptotic properties of solutions to nonautonomous ordinary differential equations. Dordrecht, Kluwer Academic Publishers Group (1993)

[2] Astashova, I. V.: Qualitative properties of solutions to quasilinear ordinary differential equations (in Russian). In: Astashova, I. V. (ed.) Qualitative Properties of Solutions to Differential Equations and Related Topics of Spectral Analysis: scientific edition, pp. 22–290. UNITY-DANA, Moscow (2012)

[3] I. V. Astashova: Asymptotic behavior of solutions of certain nonlinear differential equations. *Reports of extended session of a seminar of the I. N. Vekua Institute of Applied Mathematics* (Tbilisi), **1**:3 (1985), 9–11. (Russian)

[4] I. V. Astashova: Application of dynamical systems to the investigation of the asymptotic properties of solutions of higher-order nonlinear differential equations (in Russian), *Sovrem. Mat. Prilozh.* No. 8 (2003), 3–33; translation in *J. Math. Sci. (N.Y.)* **126** (2005), no. 5, 1361–1391.

[5] Astashova, I.: On Kiguradze's problem on power-law asymptotic behavior of blow-up solutions to Emden–Fowler type differential equations. *Georgian Mathematical Journal* **24** (2), 185–191 (2017)

[6] Astashova, I. V.: On asymptotic behavior of blow-up solutions to higher-order differential equations with general nonlinearity. In: *Differential and Difference Equations with Applications. Springer Proceedings in Mathematics and Statistics*, Vol. 230, 1–12 (2018)

[7] Kozlov, V.A.: On Kneser solutions of higher order nonlinear ordinary differential equations. *Ark. Mat.* **37**(2), 305–322 (1999)

[8] I. Astashova: On power and non-power asymptotic behavior of positive solutions to Emden-Fowler type higher-order equations, *Advances in Difference Equations. SpringerOpen Journal.* No 2013:220. 1–15 (2013)

[9] Astashova, I. V., Vasilev, M. Y.: On nonpower-law asymptotic behavior of blow-up solutions to Emden–Fowler type higher-order differential equations. In: *Differential and Difference Equations with Applications. Springer Proceedings in Mathematics and Statistics*, Vol. 333, 361–372 (2020)

[10] Astashova, I.: Atypicality of power-law solutions to Emden–Fowler type higher order equations. *St. Petersburg Mathematical Journal* **31**, 297–311 (2020)