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Blow-up of solutions of an abstract Cauchy problem for a formally hyperbolic equation with double non-linearity

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Abstract. We consider an abstract Cauchy problem for a formally hyperbolic equation with double non-linearity. Under certain conditions on the operators in the equation, we prove its local (in time) solubility and give sufficient conditions for finite-time blow-up of solutions of the corresponding abstract Cauchy problem. The proof uses a modification of a method of Levine. We give examples of Cauchy problems and initial-boundary value problems for concrete non-linear equations of mathematical physics.

Keywords: finite-time blow-up, generalized Klein–Gordon equations, non-linear hyperbolic equations, non-linear mixed boundary-value problems, field theory.

§ 1. Introduction

We shall consider an abstract Cauchy problem for a formally hyperbolic equation with double non-linearity of the following form:

$$\mathbb{A}\frac{d^2u}{dt^2}+\frac{d}{dt}\biggl(\mathbb{A}_0u+\sum_{j=1}^{n}\mathbb{A}_j(u)\biggr)+\mathbb{H}'_f(u)=\mathbb{F}'_f(u). \tag{1.1}$$

Our main concern is to find conditions for the blow-up of solutions of this problem.

First of all we note that there are three main methods for studying blow-up phenomena: the energy method of Levine [1]–[7], which was further developed in [8], the non-linear capacity method of Pokhozhaev and Mitidieri [9]–[12] and the method of self-model regimes, which is based on various comparison tests and was developed by Samarskii, Galaktionov, Kurdyumov and Mikhailov in [13], [14].

Concerning the study of Cauchy problems for equations of the form (1.1), we first of all mention the classical papers [1] and [5], which studied an abstract hyperbolic equation of the form

$$Pu_{tt}=-Au+\mathscr{F}(u)$$

and obtained sufficient conditions for finite-time blow-up of solutions of the corresponding Cauchy problem. Some initial-boundary value problems were considered as examples. Note that the Cauchy problem exhibits some difficulties for the method in [1], but they were successfully resolved in [12]. We also mention the

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classical paper [6], which studies problems for an abstract equation of formally hyperbolic type with dissipation:

$$Pu_{tt} = -Au + Bu - aPu_t + F(t,u). \tag{1.2}$$

Note that the operator acting on the first derivative with respect to time in (1.2) is linear and coincides (up to a constant factor) with the operator acting on the second derivative. Equation (1.1) shows that our situation is different.

We note that we continue to develop the modified method, which was initially applied only to Sobolev equations (see [8]).

§ 2. A differential inequality

Consider the main differential inequality

$$\Phi\Phi'' - \alpha(\Phi')^2 + \gamma\Phi'\Phi + \beta\Phi \geqslant 0, \qquad \alpha > 1, \quad \beta \geqslant 0, \quad \gamma \geqslant 0, \tag{2.1}$$

where

$$\Phi(t) \in \mathbb{C}^{(2)}([0,T]), \qquad \Phi(t) \geqslant 0, \quad \Phi(0) > 0.$$

Dividing both sides of (2.1) by $\Phi^{1+\alpha}$ and making certain calculations, we obtain the inequality

$$\left(\frac{\Phi'}{\Phi^\alpha}\right)' + \gamma\frac{\Phi'}{\Phi^\alpha} + \beta\Phi^{-\alpha} \geqslant 0,$$

which in its turn means that

$$\frac{1}{1-\alpha}(\Phi^{1-\alpha})'' + \frac{\gamma}{1-\alpha}(\Phi^{1-\alpha})' + \beta\Phi^{-\alpha} \geqslant 0. \tag{2.2}$$

We introduce the notation

$$Z(t) = \Phi^{1-\alpha}(t). \tag{2.3}$$

Using this notation, we obtain from (2.2) that

$$Z'' + \gamma Z' - \beta(\alpha-1)Z^{\alpha_1} \leqslant 0, \qquad \alpha_1 = \frac{\alpha}{\alpha-1}. \tag{2.4}$$

We also write

$$Y(t) = e^{\gamma t}Z(t). \tag{2.5}$$

This enables us to rewrite (2.4) in the form

$$Y'' - \gamma Y' - \beta(\alpha-1)e^{-\delta t}Y^{\alpha_1} \leqslant 0, \qquad \delta = \frac{\gamma}{\alpha-1}. \tag{2.6}$$

Note that we have a chain of equalities

$$Y' = (\Phi^{1-\alpha}e^{\gamma t})' = \Phi^{-\alpha}(\alpha-1)e^{\gamma t}\left[-\Phi'(t) + \frac{\gamma}{\alpha-1}\Phi(t)\right]. \tag{2.7}$$

Suppose that the following initial condition holds:

$$\Phi'(0) > \frac{\gamma}{\alpha-1}\Phi(0). \tag{2.8}$$

Then there is a $t_0 > 0$ such that

$$\Phi'(t) > \frac{\gamma}{\alpha-1}\Phi(t), \qquad t \in [0, t_0). \tag{2.9}$$

We deduce from (2.9) and (2.7) that $Y'(t) < 0$ for $t \in [0, t_0)$. Since $-\gamma Y'(t) \geqslant 0$ for $t \in [0, t_0)$, it follows from (2.6) that

$$Y'' - \beta(\alpha-1)e^{-\delta t}Y^{\alpha_1} \leqslant 0, \qquad \delta = \frac{\gamma}{\alpha-1}, \quad t \in [0, t_0). \tag{2.10}$$

We now multiply both sides of (2.10) by Y' and obtain

$$Y'Y'' - \beta(\alpha-1)e^{-\delta t}Y^{\alpha_1}Y' \geqslant 0, \qquad \delta = \frac{\gamma}{\alpha-1}, \quad t \in [0, t_0). \tag{2.11}$$

It is easy to see that the following equality holds:

$$e^{-\delta t}Y^{\alpha_1}Y' = \frac{1}{1+\alpha_1}\frac{d}{dt}[e^{-\delta t}Y^{1+\alpha_1}] + \frac{1}{1+\alpha_1}\delta e^{-\delta t}Y^{1+\alpha_1}. \tag{2.12}$$

We substitute (2.12) into (2.11) and obtain the inequality

$$Y'Y'' - \frac{\beta(\alpha-1)}{1+\alpha_1}\frac{d}{dt}[e^{-\delta t}Y^{1+\alpha_1}] - \frac{\beta(\alpha-1)\delta}{1+\alpha_1}e^{-\delta t}Y^{1+\alpha_1} \geqslant 0, \qquad t \in [0, t_0),$$

which yields that

$$Y'Y'' - \frac{\beta(\alpha-1)}{1+\alpha_1}\frac{d}{dt}[e^{-\delta t}Y^{1+\alpha_1}] \geqslant 0, \qquad t \in [0, t_0). \tag{2.13}$$

Integrating (2.13), we get

$$(Y')^2 \geqslant A^2 + \frac{2\beta(\alpha-1)^2}{2\alpha-1}e^{-\delta t}Y^{1+\alpha_1} \geqslant A^2, \tag{2.14}$$

where

$$A^2 \equiv (Y'(0))^2 - \frac{2\beta(\alpha-1)^2}{2\alpha-1}Y^{1+\alpha_1}(0). \tag{2.15}$$

We now assume that $A^2 > 0$. After some calculations, this condition takes the form

$$A^2 = (\alpha-1)^2\Phi^{-2\alpha}(0)\left[\left(\Phi'(0) - \frac{\gamma}{\alpha-1}\Phi(0)\right)^2 - \frac{2\beta}{2\alpha-1}\Phi(0)\right] > 0. \tag{2.16}$$

Hence the condition $A^2 > 0$ is equivalent to the condition

$$\left(\Phi'(0) - \frac{\gamma}{\alpha-1}\Phi(0)\right)^2 > \frac{2\beta}{2\alpha-1}\Phi(0). \tag{2.17}$$

We now conclude from (2.14) and (2.16) that

$$Y'(t) \leqslant -A < 0 \quad \Longrightarrow \quad \Phi'(t_0) > \frac{\gamma}{\alpha-1}\Phi(t_0).$$

Then $Y'(t_0) < 0$. Therefore, using the algorithm of continuation in time, we get

$$Y'(t) < 0 \quad \forall\, t \in [0,T].$$

Hence the following inequalities hold:

$$|Y'| \geqslant A > 0 \implies Y'(t) \leqslant -A \implies Y(t) \leqslant Y(0) - At$$
$$\implies \Phi^{1-\alpha}(t) \leqslant e^{-\gamma t}[\Phi^{1-\alpha}(0) - At] \implies \Phi(t) \geqslant \frac{e^{\gamma t/(\alpha-1)}}{[\Phi^{1-\alpha}(0) - At]^{1/(\alpha-1)}}.$$

We state the result as a theorem.

Theorem 2.1. *Suppose that $\Phi(t) \in \mathbb{C}^{(2)}([0,T])$ and the following conditions hold*:

$$\Phi'(0) > \frac{\gamma}{\alpha-1}\Phi(0), \tag{2.18}$$

$$\left(\Phi'(0) - \frac{\gamma}{\alpha-1}\Phi(0)\right)^2 > \frac{2\beta}{2\alpha-1}\Phi(0) \tag{2.19}$$

and, moreover, $\Phi(t) \geqslant 0$, $\Phi(0) > 0$. *Then $T > 0$ cannot be arbitrarily large. Namely, we have*

$$T \leqslant T_\infty \leqslant \Phi^{1-\alpha}(0)A^{-1},$$

$$A^2 \equiv (\alpha-1)^2\Phi^{-2\alpha}(0)\left[\left(\Phi'(0) - \frac{\gamma}{\alpha-1}\Phi(0)\right)^2 - \frac{2\beta}{2\alpha-1}\Phi(0)\right],$$

where

$$\Phi(t) \geqslant \frac{e^{\gamma t/(\alpha-1)}}{(\Phi^{1-\alpha}(0) - At)^{1/(\alpha-1)}}.$$

§ 3. Statement of the problem

The classical statement of the problem under consideration is

$$\mathbb{A}\frac{d^2u}{dt^2} + \frac{d}{dt}\left(\mathbb{A}_0 u + \sum_{j=1}^{n} \mathbb{A}_j(u)\right) + \mathbb{H}'_f(u) = \mathbb{F}'_f(u), \tag{3.1}$$

$$u(0) = u_0, \qquad u'(0) = u_1, \tag{3.2}$$

where $\mathbb{H}'_f$ and $\mathbb{F}'_f$ are the Fréchet derivatives of the corresponding functionals. To state our assumptions on the operator coefficients, we introduce some notation.

Consider Banach spaces $\mathbb{V}$, $\mathbb{V}_0$, $\mathbb{V}_j$, $\mathbb{W}_i$ for $j = 1, \dots, n$ and $i = 1, 2$ with norms $\|\cdot\|$, $\|\cdot\|_0$, $\|\cdot\|_j$, $|\cdot|_i$ respectively. Let $\mathbb{V}^*$, $\mathbb{V}_0^*$, $\mathbb{V}_j^*$, $\mathbb{W}_i^*$ be the spaces conjugate to $\mathbb{V}$, $\mathbb{V}_0$, $\mathbb{V}_j$, $\mathbb{W}_i$ with respect to the duality brackets $\langle\cdot\,,\,\cdot\rangle$, $\langle\cdot\,,\,\cdot\rangle_0$, $\langle\cdot\,,\,\cdot\rangle_j$, $(\cdot\,,\,\cdot)_i$, with norms $\|\cdot\|^*$, $\|\cdot\|_0^*$, $\|\cdot\|_j^*$, $|\cdot|_i^*$ respectively.

Suppose that the Banach spaces $\mathbb{V}_0$, $\mathbb{V}$, $\mathbb{V}_j$, $\mathbb{W}_i$ are reflexive and separable for $j = 1, \ldots, n$ and $i = 1, 2$. Suppose also that

$$\mathbb{A}\colon \mathbb{V} \to \mathbb{V}^*, \qquad \mathbb{A}_0\colon \mathbb{V}_0 \to \mathbb{V}_0^*, \qquad \mathbb{A}_j\colon \mathbb{V}_j \to \mathbb{V}_j^*.$$

Moreover, we assume that

$$\mathbb{H}(u)\colon \mathbb{W}_1 \to \mathbb{R}, \qquad \mathbb{F}(u)\colon \mathbb{W}_2 \to \mathbb{R}.$$

The Banach spaces $\mathbb{W}_1$ and $\mathbb{V}$ are assumed to be uniformly convex (this will be used in § 6).

We now state conditions on the operator-valued coefficients in (3.1).

Conditions $\mathbb{A}$.

(i) The operator $\mathbb{A}\colon \mathbb{V} \to \mathbb{V}^*$ is linear, continuous and symmetric. We have $\|\mathbb{A}u\|^* \leqslant M\|u\|$ for all $u \in \mathbb{V}$.

(ii) The operator $\mathbb{A}$ is coercive and we have $\langle \mathbb{A}u, u\rangle \geqslant \mathrm{m}\|u\|^2$ for all $u \in \mathbb{V}$.

(iii) The expression $\langle \mathbb{A}u, u\rangle^{1/2}$ is a norm on $\mathbb{V}$ inducing the original topology of the Banach space $\mathbb{V}$.

Conditions $\mathbb{A}_0$.

(i) The operator $\mathbb{A}_0\colon \mathbb{V}_0 \to \mathbb{V}_0^*$ is linear, continuous, symmetric and non-negative definite. We have $\|\mathbb{A}_0 u\|_0^* \leqslant M_0\|u\|_0$ for all $u \in \mathbb{V}_0$.

(ii)* The operator $\mathbb{A}_0$ is coercive and we have $\langle \mathbb{A}_0 u, u\rangle_0 \geqslant \mathrm{m}_0\|u\|_0^2$ for all $u \in \mathbb{V}_0$.

(iii)* The expression $\langle \mathbb{A}_0 u, u\rangle_0^{1/2}$ is a norm on $\mathbb{V}_0$ inducing the original topology of the Banach space $\mathbb{V}_0$.

Conditions $\mathbb{A}_j$.

(i) The operator $\mathbb{A}_j\colon \mathbb{V}_j \to \mathbb{V}_j^*$ is continuous.

(ii) The operator $\mathbb{A}_j$ is Fréchet differentiable and its Fréchet derivative $\mathbb{A}'_{jf}(u) \in \mathbb{C}_b(\mathbb{V}_j; \mathcal{L}(\mathbb{V}_j, \mathbb{V}_j^*))$ is a continuous, bounded, symmetric, monotone, and non-negative definite operator for a fixed $u \in \mathbb{V}_j$. We have $\mathbb{A}'_{jf}(0) = \theta$ and $\langle \mathbb{A}_j(v), v\rangle_j \geqslant 0$ for $v \in \mathbb{V}_j$.

(iii) The operator $\mathbb{A}_j$ is positive homogeneous:

$$\mathbb{A}_j(ru) = r^{p_j - 1}\mathbb{A}_j(u) \qquad \text{for} \quad p_j > 2, \quad r \geqslant 0, \quad u \in \mathbb{V}_j.$$

(iv) We have upper bounds

$$\|\mathbb{A}_j(u)\|_j^* \leqslant M_j\|u\|_j^{p_j-1}, \quad \|\mathbb{A}'_{jf}(u)\|_{\mathbb{V}_j \to \mathbb{V}_j^*} \leqslant \overline{M}_j\|u\|_j^{p_j-2}, \qquad M_j > 0.$$

(v)* We have lower bounds

$$\langle \mathbb{A}_j(u), u\rangle_j \geqslant \mathrm{m}_j\|u\|_j^{p_j}, \qquad \mathrm{m}_j > 0.$$

(vi)* The expression $\langle \mathbb{A}_j(u), u\rangle_j^{1/p_j}$ is a norm on the Banach space $\mathbb{V}_j$ inducing the original topology of $\mathbb{V}_j$.

Remark 3.1. The first estimate in condition $\mathbb{A}_j$, (iv) follows from the second. To prove this, we consider the functional $\mathbb{G}(u) = \langle \mathbb{A}_j(u), y\rangle_j$, where $y \in \mathbb{V}_j$ is an arbitrary fixed element with $\|y\|_j = 1$. Then the functional $\mathbb{G}(u) = \langle \mathbb{A}_j(u), y\rangle_j$ is Fréchet differentiable and its derivative is given by

$$\mathbb{G}'_f(u)h = \langle \mathbb{A}'_{jf}(u)h, y\rangle_j.$$

Indeed,

$$\begin{aligned}\frac{|\langle \mathbb{A}_j(u+h), y\rangle_j - \langle \mathbb{A}_j(u), y\rangle_j - \langle \mathbb{A}'_{jf}(u)h, y\rangle_j|}{\|h\|_j}&\\ &= \frac{|\langle \mathbb{A}_j(u+h) - \mathbb{A}_j(u) - \mathbb{A}'_{jf}(u)h, y\rangle_j|}{\|h\|_j}\\ &\leqslant \frac{\|\mathbb{A}_j(u+h) - \mathbb{A}_j(u) - \mathbb{A}'_{jf}(u)h\|_j^*}{\|h\|_j} \to 0\end{aligned}$$

because the operator $\mathbb{A}_j(u)$ is Fréchet differentiable. Furthermore, by condition $\mathbb{A}_j$, (iii) we have $\mathbb{A}_j(\theta_{\mathbb{V}_j}) = \theta_{\mathbb{V}_j^*}$. Then Lagrange's formula for Fréchet-differentiable functionals yields that

$$|\langle \mathbb{A}_j(u), y\rangle_j| = |\langle \mathbb{A}'_{jf}(\lambda u)u, y\rangle_j| \leqslant \overline{M}_j \|\lambda u\|_j^{p_j-2} \|u\|_j \|y\|_j, \qquad \lambda \in (0,1).$$

Taking the supremum over all $y \in \mathbb{V}_j$ with $\|y\|_j = 1$, we obtain the first estimate in conditions $\mathbb{A}_j$, (iv).

Remark 3.2. When we say that a linear operator $\mathbb{D}\colon \mathbb{B} \to \mathbb{B}^*$ is symmetric, we mean that $\langle \mathbb{D}u, v\rangle = \langle \mathbb{D}v, u\rangle$ for all $u, v \in \mathbb{B}$, where $\langle\,\cdot\,,\cdot\,\rangle$ are the duality brackets between $\mathbb{B}$ and $\mathbb{B}^*$.

Conditions $\mathbb{H}$.

(i) The functional $\mathbb{H}\colon \mathbb{W}_1 \to \mathbb{R}$ is non-negative and Fréchet differentiable. The Fréchet derivative $\mathbb{H}'_f$ is boundedly Lipschitz continuous, that is, we have

$$|\mathbb{H}'_f(u_1) - \mathbb{H}'_f(u_2)|_1^* \leqslant \mu_1(R)|u_1 - u_2|_1 \quad \forall\, u_1, u_2 \in \mathbb{W}_1,$$

where the function $\mu_1 = \mu_1(R)$ is bounded on every compact set and non-decreasing, and $R = \max\{|u_1|_1, |u_2|_1\}$.

(ii) The Fréchet derivative of $\mathbb{H}$ satisfies the upper bound

$$|\mathbb{H}'_f(u)|_1^* \leqslant M_H |u|_1^{p-1} \quad \forall\, u \in \mathbb{W}_1, \quad p \geqslant 2.$$

(iii) There is a constant $\mu > 0$ such that for all $u \in \mathbb{W}_1$ we have $(\mathbb{H}'_f(u), u)_1 \leqslant \mu\mathbb{H}(u)$.

(iv)* For all $u \in \mathbb{W}_1$ we have $c_1|u|_1^p \leqslant (\mathbb{H}'_f(u), u)_1$.

Conditions $\mathbb{F}$.

(i) The functional $\mathbb{F}\colon \mathbb{W}_2 \to \mathbb{R}$ is Fréchet differentiable. The Fréchet derivative $\mathbb{F}'_f$ is boundedly Lipschitz continuous, that is, we have

$$|\mathbb{F}'_f(u_1) - \mathbb{F}'_f(u_2)|_2^* \leqslant \mu_2(R)|u_1 - u_2|_2 \quad \forall\, u_1, u_2 \in \mathbb{W}_2,$$

where the function $\mu_2 = \mu_2(R)$ is bounded on every compact set and non-decreasing, and $R = \max\{|u_1|_2, |u_2|_2\}$.

(ii) The Fréchet derivative of $\mathbb{F}$ satisfies the upper bound

$$|\mathbb{F}'_f(u)|_2^* \leqslant M_F |u|_2^{q+1} \quad \forall\, u \in \mathbb{W}_2, \quad q > 0.$$

(iii) There is a constant $\theta > 2$ such that $\theta\mathbb{F}(u) \leqslant (\mathbb{F}'_f(u), u)_2$ for all $u \in \mathbb{W}_2$.

The starred conditions $\mathbb{A}$–$\mathbb{F}$ (for example, (iii)*) are used only in theorems on weak generalized solubility. The non-starred conditions are required for strong generalized solubility.

We now state conditions imposed on the Banach spaces $\mathbb{V}$, $\mathbb{V}_0$, $\mathbb{V}_j$, $\mathbb{W}_i$ for $j = 1, \dots, n$ and $i = 1, 2$.

Conditions $\mathbb{L}$. There is a Hilbert space $\mathbb{L}$ (identified with its conjugate) such that we have continuous embeddings

$$\mathbb{V} \subset \mathbb{L} \subset \mathbb{V}^*, \quad \mathbb{V}_j \subset \mathbb{L} \subset \mathbb{V}_j^*, \quad \mathbb{W}_i \subset \mathbb{L} \subset \mathbb{W}_i^*, \qquad j = 0, \dots, n, \quad i = 1, 2.$$

Let

$$\mathbb{W} = \mathbb{V} \cap \Bigl(\bigcap \mathbb{V}_j\Bigr) \cap \Bigl(\bigcap \mathbb{W}_i\Bigr)$$

be the reflexive separable Banach space possessing the property of density of the embeddings

$$\mathbb{W} \overset{\mathrm{ds}}{\subset} \mathbb{V} \overset{\mathrm{ds}}{\subset} \mathbb{V}^* \overset{\mathrm{ds}}{\subset} \mathbb{W}^*, \quad \mathbb{W} \overset{\mathrm{ds}}{\subset} \mathbb{V}_j \overset{\mathrm{ds}}{\subset} \mathbb{V}_j^* \overset{\mathrm{ds}}{\subset} \mathbb{W}^*, \quad \mathbb{W} \overset{\mathrm{ds}}{\subset} \mathbb{W}_i \overset{\mathrm{ds}}{\subset} \mathbb{W}_i^* \overset{\mathrm{ds}}{\subset} \mathbb{W}^*$$

for all $i = 1, 2$ and $j = 0, \dots, n$.

Definition 3.3. A *strong generalized solution of the problem* (3.1), (3.2) is a function

$$u(t) \in \mathbb{C}^{(2)}([0, T]; \mathbb{W}), \quad u(0) = u_0 \in \mathbb{W}, \quad u'(0) = u_1 \in \mathbb{W}, \tag{3.3}$$

such that for all $t \in (0, T)$ we have

$$\langle\langle D(u), w \rangle\rangle = 0 \quad \forall\, w \in \mathbb{W}, \tag{3.4}$$

where

$$D(u) = \mathbb{A}\frac{d^2u}{dt^2} + \frac{d}{dt}\biggl(\mathbb{A}_0 u + \sum_{j=1}^{n} \mathbb{A}_j(u)\biggr) + \mathbb{H}'_f(u) - \mathbb{F}'_f(u)$$

and $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle$ are the duality brackets between $\mathbb{W}$ and $\mathbb{W}^*$.

Definition 3.4. A *weak generalized solution of the problem* (3.1), (3.2) is a function

$$u(t) \in \mathbb{L}^\infty(0, T; \mathbb{W}_1), \qquad u' \in \mathbb{L}^\infty(0, T; \mathbb{V}) \cap \mathbb{L}^2(0, T; \mathbb{V}_0), \tag{3.5}$$

$$\frac{d}{dt}\mathbb{A}_j(u) \in \mathbb{L}^{p'_j}(0, T; \mathbb{V}_j^*), \quad j = 1, \dots, n, \qquad \frac{d}{dt}\mathbb{A}_0(u) \in \mathbb{L}^2(0, T; \mathbb{V}_0^*), \tag{3.6}$$

such that the equality

$$\begin{aligned} \biggl\langle\biggl\langle \frac{d}{dt}\langle \mathbb{A}u', w\rangle + \sum_{j=0}^{n} \biggl\langle \frac{d}{dt}\mathbb{A}_j(u), w \biggr\rangle_j \\ + (\mathbb{H}'_f(u), w)_1 - (\mathbb{F}'_f(u), w)_2,\ \psi(t) \biggr\rangle\biggr\rangle_{\mathcal{D}} = 0 \end{aligned} \tag{3.7}$$

holds for all $w \in \mathbb{W}_1$ and all $\psi(t) \in \mathcal{D}(0,T)$, where $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle_{\mathcal{D}}$ are the duality brackets between the space $\mathcal{D}(0,T)$ of test functions and the corresponding space $\mathcal{D}'(0,T)$ of distributions and, moreover, the following initial conditions hold: $u(0)=u_0 \in \mathbb{W}_1$, $u'(0)=u_1\in\mathbb{V}$.

§ 4. Auxiliary results

In this section, some results that are used in the main text will be proved in the required general form.

Lemma 4.1. *If an operator $\mathbb{A}\colon \mathbb{X}\to\mathbb{X}^*$ is Fréchet differentiable and has a symmetric Fréchet derivative*

$$\mathbb{A}'_u(u)\colon \mathbb{X}\to\mathcal{L}(\mathbb{X},\mathbb{X}^*),$$

and if $\mathbb{A}(su)=s^{p-1}\mathbb{A}(u)$ for all $s\geqslant 0$ and some $p\geqslant 2$, where $\mathbb{X}$ is a Banach space with conjugate $\mathbb{X}^*$ and duality brackets $\langle\,\cdot\,,\,\cdot\,\rangle$, then the functional

$$\psi(u)\equiv\langle\mathbb{A}(u),u\rangle\colon \mathbb{X}\to\mathbb{R}$$

is continuously Fréchet differentiable and its Fréchet derivative is

$$\psi'_f(u)=p\mathbb{A}(u)\quad \forall\, u\in\mathbb{X}.$$

Proof. Let us prove the operator identity

$$\mathbb{A}'_u(u)u=(p-1)\mathbb{A}(u).$$

On one hand, we have (by condition $\mathbb{A}_j$, (iii)),

$$\frac{d}{ds}\mathbb{A}(su)=\frac{d}{ds}(s^{p-1}\mathbb{A}(u))=(p-1)s^{p-2}\mathbb{A}(u)=\frac{p-1}{s}s^{p-1}\mathbb{A}(u)=\frac{p-1}{s}\mathbb{A}(su). \tag{4.1}$$

On the other hand, by the chain rule for Fréchet derivatives,

$$\frac{d}{ds}\mathbb{A}(su)=\mathbb{A}'_u(su)u. \tag{4.2}$$

Combining (4.1) and (4.2), we have

$$\frac{p-1}{s}\mathbb{A}(su)=\mathbb{A}'_u(su)u.$$

Putting $s=1$, we get the required identity,

$$(p-1)\mathbb{A}(u)=\mathbb{A}'_u(u)u\quad \forall\, u\in\mathbb{X}.$$

We make the following calculations:

$$\begin{aligned}
\psi(u+h)-\psi(u)&=\langle\mathbb{A}(u+h),u+h\rangle-\langle\mathbb{A}(u),u\rangle\\
&=\langle\mathbb{A}(u)+\mathbb{A}'_u(u)h+\omega(u,h),u+h\rangle-\langle\mathbb{A}(u),u\rangle\\
&=\langle\mathbb{A}(u),h\rangle+\langle\mathbb{A}'_u(u)h+\omega(u,h),u+h\rangle\\
&=\langle\mathbb{A}(u),h\rangle+\langle\mathbb{A}'_u(u)h,u\rangle+\langle\mathbb{A}'_u(u)h,h\rangle+\langle\omega(u,h),u+h\rangle\\
&=\langle\mathbb{A}(u)+\mathbb{A}'_u(u)u,h\rangle+\overline{\omega}(u,h),
\end{aligned}$$

where $\overline{\omega}(u,h) = \langle \mathbb{A}'_u(u)h, h\rangle + \langle \omega(u,h), u+h\rangle$. Moreover,

$$|\overline{\omega}(u,h)| \leqslant \|\mathbb{A}'_u(u)h\|_* \|h\| + \|\omega(u,h)\|_*(\|u\| + \|h\|)$$
$$\leqslant c_1\|h\|^2 + \|\omega(u,h)\|_*(\|u\| + \|h\|).$$

Finally,

$$\lim_{\|h\|\to 0} \frac{|\overline{\omega}(u,h)|}{\|h\|} = 0.$$

Hence the Fréchet derivative of the functional $\psi(u)$ is given by

$$\psi'_f(u) = \mathbb{A}(u) + \mathbb{A}'_u(u)u = \mathbb{A}(u) + (p-1)\mathbb{A}(u) = p\mathbb{A}(u).$$

It follows that $\psi(u)$ is continuously Fréchet differentiable. □

Lemma 4.2. *Under the hypotheses of Lemma 4.1 suppose that* $u(t) \in \mathbb{C}^{(1)}([0,T];\mathbb{X})$ *for some* $T > 0$. *Then*

$$\psi(u)(t) \equiv \langle \mathbb{A}(u), u\rangle \in \mathbb{C}^{(1)}([0,T]).$$

Proof. We first note that, by Lemma 4.1 and the chain rule for Fréchet derivatives,

$$\frac{d\psi}{dt} = \langle \psi'_f(u), u'\rangle = p\langle \mathbb{A}(u), u'\rangle.$$

Consider the function $f(t) \equiv \langle \mathbb{A}(u), u'\rangle$. We claim that $f(t) \in \mathbb{C}([0,T])$. Indeed, fix any $t \in [0,T]$ and suppose that $t+s \in [0,T]$. Then

$$\begin{aligned} f(t+s) - f(s) &= \langle \mathbb{A}(u(t+s)), u'(t+s)\rangle - \langle \mathbb{A}(u(t)), u'(t)\rangle \\ &= \langle \mathbb{A}(u(t)) + \mathbb{A}'_u(u(t))[u(t+s) - u(t)] + \omega(t,s), u'(t+s)\rangle \\ &\quad - \langle \mathbb{A}(u(t)), u'(t)\rangle = \langle \mathbb{A}(u(t)), u'(t+s) - u'(t)\rangle \\ &\quad + \langle \mathbb{A}'_u(u(t))\,[u(t+s) - u(t)]\,, u'(t+s)\rangle + \langle \omega(t,s), u'(t+s)\rangle. \end{aligned}$$

Using this chain of equalities, we arrive at the inequality

$$\begin{aligned} |f(t+s) - f(t)| &\leqslant \|\mathbb{A}(u(t))\|_* \|u'(t+s) - u'(t)\| \\ &\quad + \|\mathbb{A}'_u(u(t))\|_{\mathcal{L}(\mathbb{X};\mathbb{X}^*)} \|u(t+s) - u(t)\|\, \|u'(t+s)\| + \|\omega(t,s)\|_* \|u'(t+s)\|. \end{aligned}$$

Note that

$$\|u'(t+s)\| \leqslant \|u'(t)\| + \|u'(t+s) - u'(t)\| \leqslant c_1,$$

where $c_1 > 0$ is independent of $t, s \in [0,T]$. Thus we get a bound

$$|f(t+s) - f(t)| \leqslant c_2\|u'(t+s) - u'(t)\| + c_3\|u(t+s) - u(t)\| + c_4\|\omega(t,s)\|_*,$$

where $c_2, c_3, c_4 \in (0,+\infty)$ depend only on $t \in [0,T]$. Moreover,

$$\lim_{\|u(t+s)-u(t)\|\to 0} \|\omega(t,s)\|_* = 0.$$

It follows that

$$\lim_{s\to 0} f(t+s) = f(t).$$

Thus $f(t) \in \mathbb{C}([0,T])$, as required. □

Lemma 4.3. *Under the hypotheses of Lemma 4.1 suppose that* $u(t) \in \mathbb{C}^{(1)}([0,T];\mathbb{X})$ *for some* $T > 0$*. Then*

$$\langle (\mathbb{A}(u))', u\rangle = \frac{p-1}{p}\frac{d}{dt}\langle \mathbb{A}(u), u\rangle.$$

Proof. By Lemma 4.1 we have

$$\frac{d}{dt}\langle \mathbb{A}(u), u\rangle = p\langle \mathbb{A}(u), u'\rangle. \tag{4.3}$$

Moreover,

$$p\langle \mathbb{A}(u), u'\rangle = \frac{d}{dt}\langle \mathbb{A}(u), u\rangle = \langle (\mathbb{A}(u))', u\rangle + \langle \mathbb{A}(u), u'\rangle,$$

whence

$$(p-1)\langle \mathbb{A}(u), u'\rangle = \frac{d}{dt}\langle \mathbb{A}(u), u\rangle = \langle (\mathbb{A}(u))', u\rangle. \tag{4.4}$$

Comparing (4.3) and (4.4), we get the required result. □

§ 5. Blow-up of solutions

Let $T > 0$ be such that a strong generalized solution of the problem (3.1), (3.2) exists on $[0,T]$. We introduce the following notation:

$$\Phi(t) = \frac{1}{2}\langle \mathbb{A}u, u\rangle + \int_0^t \biggl(\frac{1}{2}\langle \mathbb{A}_0 u, u\rangle_0 + \sum_{j=1}^n \frac{p_j-1}{p_j}\langle \mathbb{A}_j(u), u\rangle_j\biggr)\,ds + \frac{1}{2p_0}\langle \mathbb{A}_0 u_0, u_0\rangle_0 + \frac{1}{p_0}\sum_{j=1}^n \frac{p_j-1}{p_j}\langle \mathbb{A}_j(u_0), u_0\rangle_j, \tag{5.1}$$

$$J(t) = \langle \mathbb{A}u', u'\rangle + \int_0^t \biggl(\langle \mathbb{A}_0 u', u'\rangle_0 + \sum_{j=1}^n \langle (\mathbb{A}_j(u))', u'\rangle_j\biggr)\,ds + \frac{1}{2}\langle \mathbb{A}_0 u_0, u_0\rangle_0 + \sum_{j=1}^n \frac{p_j-1}{p_j}\langle \mathbb{A}_j(u_0), u_0\rangle_j, \tag{5.2}$$

where $p_0 = \max_{j=1,\dots,n} p_j$.

Lemma 5.1. *We have*

$$(\Phi')^2 \leqslant p_0 \Phi J \quad \forall\, t \in [0,T]. \tag{5.3}$$

Proof. The following equality holds:

$$\Phi' = \langle \mathbb{A}u, u'\rangle + \frac{1}{2}\langle \mathbb{A}_0 u, u\rangle_0 + \sum_{j=1}^n \frac{p_j-1}{p_j}\langle \mathbb{A}_j(u), u\rangle_j \tag{5.4}$$

and, by the generalized Schwartz inequality, we have

$$|\langle \mathbb{A}u, u'\rangle| \leqslant \langle \mathbb{A}u, u\rangle^{1/2}\langle \mathbb{A}u', u'\rangle^{1/2}. \tag{5.5}$$

Finally, there is a chain of equalities

$$\begin{aligned}\frac{p_j-1}{p_j}\langle\mathbb{A}_j(u),u\rangle_j &= \frac{p_j-1}{p_j}\int_0^t \frac{d}{ds}\langle\mathbb{A}_j(u),u\rangle_j\,ds + \frac{p_j-1}{p_j}\langle\mathbb{A}_j(u_0),u_0\rangle_j \\ &= \int_0^t \langle(\mathbb{A}_j(u))',u\rangle_j\,ds + \frac{p_j-1}{p_j}\langle\mathbb{A}_j(u_0),u_0\rangle_j \\ &= \int_0^t \langle\mathbb{A}'_{jf}(u)u',u\rangle_j\,ds + \frac{p_j-1}{p_j}\langle\mathbb{A}_j(u_0),u_0\rangle_j, \end{aligned} \tag{5.6}$$

where we have used Lemma 4.3. We again use Schwartz' inequality and get

$$|\langle\mathbb{A}'_{jf}(u)u',u\rangle_j| \leqslant \langle\mathbb{A}'_{jf}(u)u',u'\rangle_j^{1/2}\langle\mathbb{A}'_{jf}(u)u,u\rangle_j^{1/2}. \tag{5.7}$$

Note that $\mathbb{A}'_{jf}(u)u=(p_j-1)\mathbb{A}_j(u)$. Therefore (5.7) yields the inequality

$$|\langle\mathbb{A}'_{jf}(u)u',u\rangle_j| \leqslant \langle\mathbb{A}'_{jf}(u)u',u'\rangle_j^{1/2}(p_j-1)^{1/2}\langle\mathbb{A}_j(u),u\rangle_j^{1/2}. \tag{5.8}$$

Thus it follows from (5.6) and (5.8) that

$$\begin{aligned}\frac{p_j-1}{p_j}\langle\mathbb{A}_j(u),u\rangle_j &\leqslant \int_0^t \langle\mathbb{A}'_{jf}(u)u',u'\rangle_j^{1/2}(p_j-1)^{1/2}\langle\mathbb{A}_j(u),u\rangle_j^{1/2}\,ds \\ &\qquad\qquad + \frac{p_j-1}{p_j}\langle\mathbb{A}_j(u_0),u_0\rangle_j \\ \leqslant \bigg(\int_0^t &\langle\mathbb{A}'_{jf}(u)u',u'\rangle_j\,ds\bigg)^{1/2}(p_j-1)^{1/2}\bigg(\int_0^t\langle\mathbb{A}_j(u),u\rangle_j\,ds\bigg)^{1/2} \\ &+\bigg(\frac{p_j-1}{p_j}\bigg)^{1/2}\langle\mathbb{A}_j(u_0),u_0\rangle_j^{1/2}\bigg(\frac{p_j-1}{p_j}\bigg)^{1/2}\langle\mathbb{A}_j(u_0),u_0\rangle_j^{1/2}. \end{aligned} \tag{5.9}$$

We now obtain from (5.4), (5.5) and (5.9) that

$$\begin{aligned}(\Phi')^2 \leqslant \bigg(&\langle\mathbb{A}u,u\rangle^{1/2}\langle\mathbb{A}u',u'\rangle^{1/2} \\ &+\bigg(\int_0^t\langle\mathbb{A}_0u',u'\rangle_0\,ds\bigg)^{1/2}\bigg(\int_0^t\langle\mathbb{A}_0u,u\rangle_0\,ds\bigg)^{1/2} \\ &+\sum_{j=1}^n\bigg(\int_0^t\langle\mathbb{A}'_{jf}(u)u',u'\rangle_j\,ds\bigg)^{1/2}(p_j-1)^{1/2}\bigg(\int_0^t\langle\mathbb{A}_j(u),u\rangle_j\,ds\bigg)^{1/2} \\ &+\frac{1}{\sqrt{2}}\langle\mathbb{A}_0u_0,u_0\rangle_0^{1/2}\frac{1}{\sqrt{2}}\langle\mathbb{A}_0u_0,u_0\rangle_0^{1/2} \\ &+\sum_{j=1}^n\bigg(\frac{p_j-1}{p_j}\bigg)^{1/2}\langle\mathbb{A}_j(u_0),u_0\rangle_j^{1/2}\bigg(\frac{p_j-1}{p_j}\bigg)^{1/2}\langle\mathbb{A}_j(u_0),u_0\rangle_j^{1/2}\bigg)^2. \end{aligned} \tag{5.10}$$

Note that we have

$$\bigg(\sum_{k=1}^m a_kb_k\bigg)^2 \leqslant \bigg(\sum_{k=1}^m a_k^2\bigg)\bigg(\sum_{k=1}^m b_k^2\bigg)$$

for all $a_k, b_k \geqslant 0,\ k = 1, \dots, m$. Therefore for an appropriate choice of a_k and b_k we get

$$\begin{aligned}(\Phi')^2 \leqslant \biggl(&\langle \mathbb{A}u, u\rangle + \int_0^t \biggl[\langle \mathbb{A}_0 u, u\rangle_0 + \sum_{j=1}^n (p_j - 1)\langle \mathbb{A}_j(u), u\rangle_j\biggr] ds \\ &+ \frac{1}{2}\langle \mathbb{A}_0 u_0, u_0\rangle_0 + \sum_{j=1}^n \frac{p_j - 1}{p_j}\langle \mathbb{A}_j(u_0), u_0\rangle_j\biggr) \\ &\times \biggl(\langle \mathbb{A}u', u'\rangle + \int_0^t \biggl[\langle \mathbb{A}_0 u', u'\rangle_0 + \sum_{j=1}^n \langle (\mathbb{A}_j(u))', u'\rangle_j\biggr] ds \\ &+ \frac{1}{2}\langle \mathbb{A}_0 u_0, u_0\rangle_0 + \sum_{j=1}^n \frac{p_j - 1}{p_j}\langle \mathbb{A}_j(u_0), u_0\rangle_j\biggr) \leqslant p_0 \Phi(t) J(t), \end{aligned} \tag{5.11}$$

where we have used the notation (5.1) and (5.2). □

We now proceed to deduce the first and second energy equalities. To do this, we put $w = u$ in (3.4) and integrate by parts:

$$\begin{aligned}&\frac{1}{2}\frac{d^2}{dt^2}\langle \mathbb{A}u, u\rangle - \langle \mathbb{A}u', u'\rangle \\ &\quad + \frac{d}{dt}\biggl(\frac{1}{2}\langle \mathbb{A}_0 u, u\rangle + \sum_{j=1}^n \frac{p_j - 1}{p_j}\langle \mathbb{A}_j(u), u\rangle_j\biggr) + (\mathbb{H}'_f(u), u)_1 = (\mathbb{F}'_f(u), u)_2, \end{aligned} \tag{5.12}$$

whence by the definition of the functional $\Phi(t)$ we obtain the first energy equality

$$\frac{d^2\Phi}{dt^2} + (\mathbb{H}'_f(u), u)_1 = \langle \mathbb{A}u', u'\rangle + (\mathbb{F}'_f(u), u)_2. \tag{5.13}$$

We now put $w = u'$ in (3.4) and integrate by parts:

$$\frac{d}{dt}\biggl[\frac{1}{2}\langle \mathbb{A}u', u'\rangle + \int_0^t \biggl(\langle \mathbb{A}_0 u', u'\rangle_0 + \sum_{j=1}^n \langle (\mathbb{A}_j(u))', u'\rangle_j\biggr) ds + \mathbb{H}(u)\biggr] = \frac{d}{dt}\mathbb{F}(u). \tag{5.14}$$

Integrating (5.14) with respect to time, we see that

$$\frac{1}{2}\langle \mathbb{A}u', u'\rangle + \int_0^t \biggl(\langle \mathbb{A}_0 u', u'\rangle_0 + \sum_{j=1}^n \langle (\mathbb{A}_j(u))', u'\rangle_j\biggr) ds - \overline{E}(0) + \mathbb{H}(u) = \mathbb{F}(u), \tag{5.15}$$

where

$$\overline{E}(0) = \frac{1}{2}\langle \mathbb{A}u_1, u_1\rangle + \mathbb{H}(u_0) - \mathbb{F}(u_0).$$

By condition $\mathbb{F}$, (iii) we have $\theta \mathbb{F}(u) \leqslant (\mathbb{F}'_f(u), u)_2$. Therefore (5.13) and (5.15) yield the inequality

$$\begin{aligned}\frac{d^2\Phi}{dt^2} + (\mathbb{H}'_f(u), u)_1 &\geqslant \langle \mathbb{A}u', u'\rangle + \frac{\theta}{2}\langle \mathbb{A}u', u'\rangle \\ &+ \theta \int_0^t \biggl(\langle \mathbb{A}_0 u', u'\rangle_0 + \sum_{j=1}^n \langle (\mathbb{A}_j(u))', u'\rangle_j\biggr) ds - \theta \overline{E}(0) + \theta \mathbb{H}(u), \end{aligned} \tag{5.16}$$

whence in view of the notation (5.2) we get

$$\frac{d^2\Phi}{dt^2}+\theta\overline{E}(0)\geqslant\theta\mathbb{H}(u)-(\mathbb{H}'_f(u),u)_1+\left(1+\frac{\theta}{2}\right)J \\ -\left(1+\frac{\theta}{2}\right)\left(\frac{1}{2}\langle\mathbb{A}_0u_0,u_0\rangle_0+\sum_{j=1}^{n}\frac{p_j-1}{p_j}\langle\mathbb{A}_j(u_0),u_0\rangle_j\right). \tag{5.17}$$

We now observe that by conditions $\mathbb{H}$, (i), (iii) we have

$$(\mathbb{H}'_f(u),u)_1\leqslant\mu\mathbb{H}(u),\qquad \mathbb{H}(u)\geqslant 0.$$

Therefore it follows from (5.17) under the assumption $\theta\geqslant\mu$ that

$$\frac{d^2\Phi}{dt^2}+E(0)\geqslant\left(1+\frac{\theta}{2}\right)J, \tag{5.18}$$

where

$$E(0)=\frac{\theta}{2}\langle\mathbb{A}u_1,u_1\rangle+\theta\mathbb{H}(u_0) \\ +\left(1+\frac{\theta}{2}\right)\left(\frac{1}{2}\langle\mathbb{A}_0u_0,u_0\rangle_0+\sum_{j=1}^{n}\frac{p_j-1}{p_j}\langle\mathbb{A}_j(u_0),u_0\rangle_j\right)-\theta\mathbb{F}(u_0). \tag{5.19}$$

Therefore we obtain the following inequality from (5.3) and (5.18):

$$\Phi\Phi''-\alpha(\Phi')^2+\beta\Phi\geqslant 0, \tag{5.20}$$

where

$$\alpha=\frac{1}{p_0}\left(1+\frac{\theta}{2}\right),\qquad \beta=E(0).$$

Thus by Theorem 2.1 we arrive at the following blow-up result.

Theorem 5.2. *Suppose that*

$$\Phi'(0)>(\delta\Phi(0))^{1/2},\quad \Phi(0)>0,\quad \theta>2(p_0-1),\qquad \theta\geqslant\mu, \tag{5.21}$$

$$\delta=\begin{cases}\dfrac{2E(0)}{2\alpha-1} & \textit{for}\quad E(0)>0,\\ 0 & \textit{for}\quad E(0)\leqslant 0,\end{cases}\qquad \alpha=\frac{1}{p_0}\left(1+\frac{\theta}{2}\right),\qquad p_0=\max_{j=1,\dots,n}p_j.$$

Then $T>0$ cannot be arbitrarily large. Namely, we have

$$T\leqslant T_\infty\leqslant\Phi^{1-\alpha}(0)A^{-1}, \\ A^2\equiv(\alpha-1)^2\Phi^{-2\alpha}(0)[(\Phi'(0))^2-\delta\Phi(0)],$$

and

$$\Phi(t)\geqslant\frac{1}{(\Phi^{1-\alpha}(0)-At)^{1/(\alpha-1)}},$$

where

$$\Phi(t) = \frac{1}{2}\langle \mathbb{A}u, u\rangle + \int_0^t \left(\frac{1}{2}\langle \mathbb{A}_0 u, u\rangle_0 + \sum_{j=1}^n \frac{p_j - 1}{p_j}\langle \mathbb{A}_j(u), u\rangle_j\right) ds$$
$$+ \frac{1}{2p_0}\langle \mathbb{A}_0 u_0, u_0\rangle_0 + \frac{1}{p_0}\sum_{j=1}^n \frac{p_j - 1}{p_j}\langle \mathbb{A}_j(u_0), u_0\rangle_j,$$

$$\Phi(0) = \frac{1}{2}\langle \mathbb{A}u_0, u_0\rangle + \frac{1}{2p_0}\langle \mathbb{A}_0 u_0, u_0\rangle_0 + \frac{1}{p_0}\sum_{j=1}^n \frac{p_j - 1}{p_j}\langle \mathbb{A}_j(u_0), u_0\rangle_j,$$

$$\Phi'(0) = \langle \mathbb{A}u_0, u_1\rangle + \frac{1}{2}\langle \mathbb{A}_0 u_0, u_0\rangle_0 + \sum_{j=1}^n \frac{p_j - 1}{p_j}\langle \mathbb{A}_j(u_0), u_0\rangle_j,$$

$$E(0) = \frac{\theta}{2}\langle \mathbb{A}u_1, u_1\rangle + \theta\mathbb{H}(u_0)$$
$$+ \left(1 + \frac{\theta}{2}\right)\left(\frac{1}{2}\langle \mathbb{A}_0 u_0, u_0\rangle_0 + \sum_{j=1}^n \frac{p_j - 1}{p_j}\langle \mathbb{A}_j(u_0), u_0\rangle_j\right) - \theta\mathbb{F}(u_0).$$

Remark 5.3. The hypotheses of Theorem 5.2 impose no restrictions on the number $E(0) > 0$. This is essential since, as a rule, the requirement that the initial energy be positive, also imposes two further conditions. Therefore our problem is to prove the consistency of the resulting hypotheses of the theorem.

Assertion 5.4. *The hypotheses of Theorem 5.2 are consistent.*

Proof. Clearly, the consistency must be verified only in the case when $E(0) > 0$. By the conditions on the operators $\mathbb{A}$, $\mathbb{A}_0$, $\mathbb{A}_j$, $j = 1, \dots, n$, we have $\Phi'(0) > 0$ for all (u_0, u_1) provided only that $u_0 \neq 0$. Hence the hypothesis (5.21) will hold if $u_0 \neq 0$ and

$$0 < \frac{\delta\Phi(0)}{(\Phi'(0))^2} < 1. \tag{5.22}$$

Clearly, to prove the consistency, it suffices to consider the case when $n = 1$. In this case, $p_0 = p_1$. We now adopt the following scheme.

1) Fix the operators $\mathbb{A}$, $\mathbb{A}_0$, $\mathbb{A}_1$.

2) Put $H(u) = |u|_1^p$, $p = \mu \geqslant 2$.

3) Choose any $\theta \geqslant \mu = p$, $\theta > 2(p_1 - 1)$. Note that then $\theta > p_1 + p_1 - 2 > p_1$ because $p_1 > 2$.

4) Put $F(u) = |u|_2^r$, where $r > \theta$. Then Steps 1–3 yield that, first, condition $\mathbb{F}$, (iii) holds, second, $r > p = \mu$, whence the order of growth of the functional $\mathbb{F}(u)$ is higher than that of $\mathbb{H}(u)$ and, third, $r > p_1$ (this will be used below).

5) Fix any non-zero u_0, u_1. Introduce the notation

$$\Psi(R) = \Phi[Ru_0, Ru_1](0), \qquad \Psi_1(R) = \Phi'[Ru_0, Ru_1](0), \qquad \delta(R) = \delta[Ru_0, Ru_1].$$

Consider the following function of R:

$$\Gamma(R) = \frac{\delta(R)\Psi(R)}{\Psi_1^2(R)}.$$

Writing it out explicitly and factoring out an appropriate power of R in each factor, we obtain the following explicit expression for $\Gamma(R)$ if we use the linearity of linear operators and condition $\mathbb{A}_1$(iii) along with the explicit forms of $\mathbb{F}(u)$ and $\mathbb{H}(u)$ and divide by R^2:

$$\Gamma(R)=\frac{\frac{1}{2}\langle\mathbb{A}u_0,u_0\rangle+\frac{1}{2p_1}\langle\mathbb{A}_0u_0,u_0\rangle+R^{p_1-2}\frac{p_1-1}{p_1^2}\langle\mathbb{A}_1(u_0),u_0\rangle_1}{\frac{1}{p_1}(2+\theta)-1}\\ \times\frac{2E(0)}{\bigl[\langle\mathbb{A}u_0,u_1\rangle+\frac{1}{2}\langle\mathbb{A}_0u_0,u_0\rangle_0+R^{p_1-2}\frac{p_1-1}{p_1}\langle\mathbb{A}_1(u_0),u_0\rangle_1\bigr]^2}, \tag{5.23}$$

where

$$2E(0)=\theta\langle\mathbb{A}u_1,u_1\rangle+(2+\theta)\biggl(\frac{1}{2}\langle\mathbb{A}_0u_0,u_0\rangle_0+R^{p_1-2}\frac{p_1-1}{p_1}\langle\mathbb{A}_1u_0,u_0\rangle_1\biggr)\\ +2\theta(R^{p-2}\mathbb{H}(u_0)-R^{r-2}\mathbb{F}(u_0)). \tag{5.24}$$

The numerator $2E(0)$ of the second quotient in (5.23) is positive since $E(0)>0$. Note that the first numerator and denominator in (5.23) are positive by the conditions imposed on the operators. We need to show that

$$0<\Gamma(R)<1. \tag{5.25}$$

We easily see that $\Gamma(R)$ tends to a positive limit Γ_0 as $R\to+0$, and that Γ_0 is determined only by the linear operators (we recall that u_0, u_1 are fixed). It is also clear that the limit of $\Gamma(R)$ as $R\to+\infty$ is equal to $-\infty$ because the highest growth exponent has a negative summand $-2\theta R^{r-2}\mathbb{F}(u_0)$ in (5.24). Since the denominator never vanishes on the interval $R\in(0,+\infty)$, the function $\Gamma(R)$ is continuous and, therefore, assumes all values in the interval $(-\infty,\Gamma_0)$, including arbitrarily small positive values. This guarantees that (5.25) holds. The condition $E(0)>0$ also holds since otherwise we would have $\Gamma(R)<0$. □

§ 6. Local solubility in the sense of weak generalized solutions

We prove the local solubility of the Cauchy problem (3.1), (3.2) in several steps, using Galerkin's method along with the methods of compactness and monotonicity.

Step 1. *Statement of the problem for Galerkin approximants.* Let $\{w_k\}$ be a Galerkin basis in the separable Hilbert space $\overline{\mathbb{W}}\overset{\mathrm{ds}}{\subset}\mathbb{W}$. We introduce the notation

$$u_m=\sum_{l=1}^{m}c_{ml}(t)w_l,\qquad c_{ml}\in\mathbb{C}^{(2)}([0,T_m]),\quad T_m>0.$$

The function u_m is called a *Galerkin approximant* to the exact solution u of the problem (3.3), (3.4) if it is a classical solution of the following system of ordinary differential equations:

$$\sum_{l=1}^{n_1} \langle\langle f_l, w_k\rangle\rangle_{l\overline{\mathbb{W}}} = 0, \qquad k = 1, \dots, m, \quad n_1 = 4 + n, \tag{6.1}$$

$$f_1 = \frac{d}{dt}\mathbb{A}\frac{du_m}{dt}, \qquad f_2 = \frac{d}{dt}\mathbb{A}_0 u_m,$$

$$f_j = \frac{d}{dt}\mathbb{A}_{j-2}(u_m),\ j = 3, \dots, 2+n, \quad f_{3+n} = \mathbb{H}'_f(u_m), \quad f_{4+n} = -\mathbb{F}'_f(u_m),$$

where the choice of the various duality brackets $\langle\langle f_l, w_k\rangle\rangle_{l\overline{\mathbb{W}}}$ between the Hilbert spaces $\overline{\mathbb{W}}$ and $\overline{\mathbb{W}}^*$ is described by the equalities (arising from the corresponding dense embeddings of $\overline{\mathbb{W}}$ in the spaces $\mathbb{V}$, $\mathbb{V}_0$, $\mathbb{V}_j$, $\mathbb{W}_1$ and $\mathbb{W}_2$):

$$\langle\langle f_1, w_k\rangle\rangle_{1\overline{\mathbb{W}}} = \langle f_1, w_k\rangle, \qquad \langle\langle f_2, w_k\rangle\rangle_{2\overline{\mathbb{W}}} = \langle f_2, w_k\rangle_0,$$
$$\langle\langle f_{3+j}, w_k\rangle\rangle_{(3+j)\overline{\mathbb{W}}} = \langle f_{3+j}, w_k\rangle_j, \qquad \langle\langle f_{4+n}, w_k\rangle\rangle_{(4+n)\overline{\mathbb{W}}} = (f_{4+n}, w_k)_1,$$
$$\langle\langle f_{5+n}, w_k\rangle\rangle_{(5+n)\overline{\mathbb{W}}} = (f_{5+n}, w_k)_2.$$

We also add the initial conditions

$$u_{m0} = u_m(0) = \sum_{l=1}^{m} c_{ml}(0) w_l \to u_0 \quad \text{strongly in } \ \mathbb{W}_1, \tag{6.2}$$

$$u_{m1} = u'_m(0) = \sum_{l=1}^{m} c'_{ml}(0) w_l \to u_1 \quad \text{strongly in } \ \mathbb{V}. \tag{6.3}$$

Step 2. *Local solubility for Galerkin approximants.* We rewrite the system (6.1)–(6.3) for Galerkin approximants in the form

$$\sum_{l=1}^{m} \langle \mathbb{A} w_l, w_k\rangle \frac{d^2 c_{ml}}{dt^2} + \frac{dc_{ml}}{dt}\biggl(\sum_{l=1}^{m} \langle \mathbb{A}_0 w_l, w_k\rangle_0 + \sum_{l=1}^{m}\sum_{j=1}^{n} \langle \mathbb{A}'_{jf}(u_m) w_l, w_k\rangle_j\biggr)$$
$$+ (\mathbb{H}'_f(u_m), w_k)_1 - (\mathbb{F}'_f(u_m), w_k)_2 = 0 \tag{6.4}$$

for $k = 1, \dots, m$. Consider the quadratic form generated by the matrix with entries

$$a_{kl} = \langle \mathbb{A} w_l, w_k\rangle.$$

By conditions $\mathbb{A}$, (ii) we have the following chain of equalities:

$$\sum_{k,j=1}^{m} a_{kj}\xi^k\xi^j = \langle \mathbb{A}\xi, \xi\rangle \geqslant m\|\xi\|^2, \qquad \xi = \sum_{k=1}^{m} \xi^k w_k.$$

Using the embedding $\mathbb{W} \subset \mathbb{V}$, we conclude from this that the matrix (a_{kj}) of our quadratic form is non-degenerate by Sylvester's criterion. Hence the matrix in front of the second derivative with respect to time in (6.4) is invertible. By the conditions introduced above, we have

$$\mathbb{A}'_{jf}(\,\cdot\,) \in \mathbb{C}(\mathbb{V}_j; \mathcal{L}(\mathbb{V}_j, \mathbb{V}_j^*)), \quad \mathbb{H}'_f(\,\cdot\,) \in \mathbb{C}(\mathbb{W}_1; \mathbb{W}_1^*), \quad \mathbb{F}'_f(\,\cdot\,) \in \mathbb{C}(\mathbb{W}_2; \mathbb{W}_2^*).$$

Therefore, by Peano's theorem, the system of ordinary differential equations (6.4) with initial conditions (6.2) and (6.3) has at least one classical solution $c_{ml}(t) \in \mathbb{C}^{(2)}([0,T_m])$ for some $T_m > 0$. Generally speaking, the number T_m depends on $m \in \mathbb{N}$. But *a priori* estimates (see below) will enable us to prove that $c_{mk}(t) \in \mathbb{C}^{(2)}([0,T])$ for some $T > 0$, where T is independent of m.

Step 3. *A priori estimates.* To deduce these, we multiply both sides of (6.1) by $c'_{mk}(t)$ and sum over $k = 1, \dots, m$ to obtain the equality

$$\frac{d}{dt}\left[\frac{1}{2}\langle \mathbb{A}u'_m, u'_m\rangle + \mathbb{H}(u_m)\right] + \langle \mathbb{A}_0 u'_m, u'_m\rangle_0 + \sum_{j=1}^{n}\langle \mathbb{A}'_{jf}(u_m)u'_m, u'_m\rangle_j = (\mathbb{F}'_f(u_m), u'_m)_2. \tag{6.5}$$

Note that by conditions $\mathbb{A}_j$ we have

$$\langle \mathbb{A}'_{jf}(u_m)u'_m, u'_m\rangle_j \geqslant 0, \qquad j = 0, \dots, n.$$

Integrating (6.5) with respect to time, we obtain

$$\frac{1}{2}\langle \mathbb{A}u'_m, u'_m\rangle + \mathbb{H}(u_m) + \int_0^t \langle \mathbb{A}_0 u'_m, u'_m\rangle_0 \, ds + \sum_{j=1}^{n}\int_0^t \langle \mathbb{A}'_{jf}(u_m)u'_m, u'_m\rangle_j \, ds = E_m(0) + \int_0^t (\mathbb{F}'_f(u_m), u'_m)_2 \, ds, \tag{6.6}$$

$$E_m(0) = \frac{1}{2}\langle \mathbb{A}u_{1m}, u_{1m}\rangle + \mathbb{H}(u_{m0}).$$

Suppose that we have

$$|(\mathbb{F}'_f(v), w)_2| \leqslant c_2 \|w\| |v|_1^{q+1} \tag{6.7}$$

and continuous embeddings

$$v, w \in \mathbb{W}_1 \subset \mathbb{W}_2 \subset \mathbb{V}. \tag{6.8}$$

Remark 6.1. Let us give an example of spaces $\mathbb{W}_1$, $\mathbb{W}_2$, $\mathbb{V}$ and an operator $\mathbb{F}$ for which inequality (6.7) holds. We put

$$\mathbb{W}_1 = \mathbb{W}_0^{1,p}(\Omega), \qquad \mathbb{W}_2 = \mathbb{L}^{q+2}(\Omega), \qquad \mathbb{V} = \mathbb{L}^2(\Omega),$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with sufficiently smooth boundary, and the functional $\mathbb{F}(u)$ is given by

$$\mathbb{F}(u) = \frac{1}{q+2}\int_\Omega |u|^{q+2} \, dx.$$

Clearly,

$$(\mathbb{F}'_f(v), w)_2 = \int_\Omega |v|^q v w \, dx.$$

Assuming that $2q+2 \leqslant p^*$, we have a chain of inequalities

$$|(\mathbb{F}'_f(v), w)_2| \leqslant \left(\int_\Omega |v|^{2q+2}\,dx\right)^{1/2}\left(\int_\Omega |w|^2\,dx\right)^{1/2} \leqslant c_2 \||\nabla v|\|_{\mathbb{L}^p}^{q+1}\|w\|_{\mathbb{L}^2}.$$

Thus (6.7) holds in this case.

Hence it follows from (6.6) and (6.7) that

$$\frac{1}{2}\langle \mathbb{A}u'_m, u'_m\rangle + \mathbb{H}(u_m) + \int_0^t \langle \mathbb{A}_0 u'_m, u'_m\rangle_0\,ds + \sum_{j=1}^n \int_0^t \langle \mathbb{A}'_{jf}(u_m)u'_m, u'_m\rangle_j\,ds \\ \leqslant E_m(0) + c_2\int_0^t \|u'_m\|\,|u_m|_1^{q+1}\,ds, \tag{6.9}$$

and by conditions $\mathbb{A}$ and $\mathbb{H}$ we have

$$\langle \mathbb{A}u'_m, u'_m\rangle \geqslant \mathrm{m}\|u'_m\|^2, \qquad \mathbb{H}(u_m) \geqslant \frac{c_1}{\mu}|u_m|_1^p.$$

Redenoting $c_1 := c_1/\mu$ and using this and (6.9), we arrive at the inequality

$$\frac{\mathrm{m}}{2}\|u'_m\|^2 + c_1|u_m|_1^p + \int_0^t \langle \mathbb{A}_0 u'_m, u'_m\rangle_0\,ds + \sum_{j=1}^n \int_0^t \langle \mathbb{A}'_{jf}(u_m)u'_m, u'_m\rangle_j\,ds \\ \leqslant E_m(0) + \frac{c_2}{2}\int_0^t [\|u'_m\|^2 + |u_m|_1^{2q+2}]\,ds. \tag{6.10}$$

Arguing in the standard way using Young's arithmetical inequality

$$ab \leqslant \frac{a^{q_1}}{q_1} + \frac{b^{q_2}}{q_2}, \qquad \frac{1}{q_1} + \frac{1}{q_2} = 1,$$

and the Gronwall–Bellman–Bihari inequality in the two cases $p \geqslant 2q+2$ and $p < 2q+2$, we conclude that the following *a priori* estimates hold:

$$\|u'_m\| \leqslant c_3(T), \quad |u_m|_1 \leqslant c_4(T), \quad \int_0^T \|u'_m\|_0^2\,dt \leqslant c_5(T), \tag{6.11}$$

$$\int_0^T \langle \mathbb{A}'_{jf}(u_m)u'_m, u'_m\rangle_j\,dt \leqslant c_6(T) \tag{6.12}$$

for all $t \in [0,T)$, where the constants $c_3,\dots,c_6$ are independent of $m \in \mathbb{N}$ and we have $T = +\infty$ when $p \geqslant 2q+2$, and $T < +\infty$ when $p < 2q+2$. Note that by conditions $\mathbb{H}$ we have

$$|\mathbb{H}'_f(v)|_1^* \leqslant M_H|v|_1^{p-1} \quad \forall\, v \in \mathbb{W}_1, \quad p \geqslant 2.$$

Combining this with the *a priori* estimates (6.11), we get another *a priori* estimate:

$$|\mathbb{H}'_f(u_m)|_1^* \leqslant c_7(T) \quad \forall\, t \in [0,T), \tag{6.13}$$

where, as above, we have either $T = +\infty$ or $T < +\infty$ depending on which case we are in.

We now aim at obtaining a second-order *a priori* estimate for the sequence $\{u''_m\}$. To do this, we choose the Galerkin basis $\{w_j\} \subset \mathbb{W}$ in a more concrete way. Suppose that the eigenvalue/eigenfunction problem

$$\mathbb{A}w_k = \lambda_k w_k$$

has a countable set of linearly independent solutions $\{w_k\} \subset \mathbb{V}$. Now suppose that $\{w_k\}$ is an orthonormal basis in $\overline{\mathbb{W}}$. We introduce the projector onto the linear span (to be denoted by $\overline{\mathbb{W}}_m$) of the elements $\{w_k\}_{k=1}^m$:

$$\mathbb{P}_m z = \sum_{k=1}^{m} (z, w_k)_{\overline{\mathbb{W}}} w_k.$$

Clearly, $\mathbb{P}_m w_k = w_k$ for $k = 1, \dots, m$. It is easy to see that

$$\mathbb{P}_m \mathbb{A} z = \mathbb{A}\mathbb{P}_m z = \mathbb{A}z, \qquad z \in \overline{\mathbb{W}}_m.$$

Using the projector $\mathbb{P}_m$, we rewrite (6.1) in the form

$$\sum_{l=1}^{n_1} \langle\langle f_l, \mathbb{P}_m z\rangle\rangle_{l\overline{\mathbb{W}}} = 0, \qquad z \in \overline{\mathbb{W}},$$

which in its turn yields that

$$\sum_{l=1}^{n_1} \langle\langle \mathbb{P}_m^{\mathrm{t}} f_l, z\rangle\rangle_{l\overline{\mathbb{W}}} = 0, \qquad z \in \overline{\mathbb{W}},$$

where the transposed operator $\mathbb{P}_m^{\mathrm{t}}$ and the conjugate operator $\mathbb{P}_m^*$ are related by the standard formula

$$\mathbb{P}_m^* = \mathbb{J}\mathbb{P}_m^{\mathrm{t}}\mathbb{J}^{-1},$$

$\mathbb{J}\colon \overline{\mathbb{W}}^* \to \overline{\mathbb{W}}$ being the (linear, continuous and invertible) Riesz–Fréchet embedding operator. Hence we have

$$\mathbb{P}_m^{\mathrm{t}} D(u_m) = \vartheta \in \overline{\mathbb{W}}^*.$$

As above, it is easy to see that

$$\mathbb{P}_m^* \mathbb{A} z = \mathbb{P}_m \mathbb{A} z, \qquad z \in \overline{\mathbb{W}}_m.$$

We now obtain sufficient conditions for the following equality to hold:

$$\mathbb{P}_m^{\mathrm{t}} \mathbb{A} z = \mathbb{P}_m \mathbb{A} z, \qquad z \in \overline{\mathbb{W}}_m.$$

Assume that the operators $\mathbb{P}_m$ and $\mathbb{J}$ commute on the space $\overline{\mathbb{W}}_m$:

$$\mathbb{P}_m \mathbb{J} w_k = \mathbb{J}\mathbb{P}_m w_k, \qquad k = 1, \dots, m. \tag{6.14}$$

Remark 6.2. We now give examples of spaces and operators for which (6.14) holds.

1) Suppose that $\overline{\mathbb{W}} = \mathbb{H}^1_0(\Omega)$, $\mathbb{A} = \mathbb{I}$,

$$-\Delta w_k = \lambda_k w_k, \qquad w_k \in \mathbb{H}^1_0(\Omega).$$

Then

$$\mathbb{J} = (-\Delta)^{-1} \colon \mathbb{H}^{-1}(\Omega) \to \mathbb{H}^1_0(\Omega).$$

Clearly, $\mathbb{J} w_k = \frac{w_k}{\lambda_k}$, $\mathbb{P}_m w_k = w_k$. Therefore the commutation formula (6.14) holds.

2) Suppose that $\overline{\mathbb{W}} = \mathbb{H}^2_0(\Omega)$, $\mathbb{A} = \mathbb{I}$,

$$\Delta^2 w_k = \lambda_k w_k, \qquad w_k \in \mathbb{H}^2_0(\Omega).$$

Then

$$\mathbb{J} = (\Delta)^{-2} \colon \mathbb{H}^{-2}(\Omega) \to \mathbb{H}^2_0(\Omega).$$

Clearly, $\mathbb{J} w_k = \frac{w_k}{\lambda_k}$, $\mathbb{P}_m w_k = w_k$. Hence the commutation formula (6.14) also holds in this case.

As a result, we arrive at the operator equality

$$\left\langle\!\left\langle \frac{d}{dt} \mathbb{A} \frac{du_m}{dt}, z \right\rangle\!\right\rangle_{1\overline{\mathbb{W}}} = -\sum_{l=2}^{n_1} \langle\langle \mathbb{P}^{\mathrm{t}}_m f_l, z\rangle\rangle_{l\overline{\mathbb{W}}}. \tag{6.15}$$

To get the desired second-order *a priori* estimate, it suffices to take the supremum of both sides of (6.15) over $z \in \overline{\mathbb{W}}$, $\|z\|_{\overline{\mathbb{W}}} = 1$. The uniform boundedness of $\mathbb{P}_m$ with respect to $m \in \mathbb{N}$ implies that the transposed operators

$$\mathbb{P}^{\mathrm{t}}_m \colon \overline{\mathbb{W}}^* \to \overline{\mathbb{W}}^*$$

are also uniformly bounded with respect to $m \in \mathbb{N}$. By the conditions on the operators stated above and the *a priori* estimates (6.11) we have the inequalities

$$\int_0^T (\|\mathbb{A}_0 u'_m\|^*_0)^2 \, dt \leqslant M_0^2 \int_0^T \|u'_m\|_0^2 \, dt \leqslant c_7(T). \tag{6.16}$$

We now note that if $\mathbb{W}_1 \subset \mathbb{V}_j$ for $j = 1, \dots, n$, then we have

$$\begin{aligned}
\|\mathbb{A}'_{jf}(u_m) u'_m\|^*_j &= \sup_{\|v\|_j = 1} |\langle \mathbb{A}'_{jf}(u_m) u'_m, v\rangle_j| \\
&\leqslant \sup_{\|v\|_j = 1} \langle \mathbb{A}'_{jf}(u_m) u'_m, u'_m\rangle_j^{1/2} \langle \mathbb{A}'_{jf}(u_m) v, v\rangle_j^{1/2} \\
&\leqslant \overline{M}_j \|u_m\|_j^{p_j/2-1} \langle \mathbb{A}'_{jf}(u_m) u'_m, u'_m\rangle_j^{1/2} \sup_{\|v\|_j = 1} \|v\|_j \\
&\leqslant \overline{M}_j c_j |u_m|_1^{p_j/2-1} \langle \mathbb{A}'_{jf}(u_m) u'_m, u'_m\rangle_j^{1/2}.
\end{aligned} \tag{6.17}$$

By the *a priori* estimates (6.11) and (6.12), we obtain from (6.17) the *a priori* estimate

$$\int_0^T (\|\mathbb{A}'_{jf}(u_m) u'_m\|^*_j)^2 \, dt \leqslant c_8(T). \tag{6.18}$$

Finally, the following *a priori* estimates hold:

$$|\mathbb{H}_f'(u_m)|_1^* \leqslant M_H|u_m|_1^{p-1} \leqslant c_9(T), \tag{6.19}$$

$$|\mathbb{F}_f'(u_m)|_2^* \leqslant M_F|u_m|_2^{q+1} \leqslant c_{10}|u_m|_1^{q+1} \leqslant c_{11}(T) \tag{6.20}$$

because $\mathbb{W}_1 \subset \mathbb{W}_2$. Thus, using the *a priori* estimates (6.16)–(6.20), we obtain from (6.15) the second-order *a priori* estimate

$$\int_0^T \left(\left\|\frac{d}{dt}\mathbb{A}u_m'\right\|_{\overline{\mathbb{W}}}^*\right)^2 dt \leqslant c_{12}(T), \tag{6.21}$$

where either $T = +\infty$ or $T < +\infty$ (depending on the conditions). We now suppose that

$$\|\mathbb{A}v\|_{\mathbb{B}} \leqslant a\langle \mathbb{A}_0 v, v\rangle_0^{1/2}, \qquad v \in \mathbb{V}_0. \tag{6.22}$$

Remark 6.3. A model realization of the Hilbert space $\mathbb{B}$ is given by the Hilbert space $\mathbb{H}_0^1(\Omega)$. Here, for example,

$$\mathbb{A} = \mathbb{I}, \qquad \mathbb{A}_0 = -\Delta, \qquad \mathbb{V}_0 = \mathbb{H}_0^1(\Omega),$$

and the inequality (6.22) takes the form

$$\|v\|_{\mathbb{H}_0^1(\Omega)} \leqslant a\||\nabla v|\|_{\mathbb{L}^2(\Omega)}, \qquad v \in \mathbb{H}_0^1(\Omega).$$

By the *a priori* estimate (6.11) we arrive at the *a priori* estimate

$$\int_0^T \|\mathbb{A}u_m'\|_{\mathbb{B}}^2\, dt \leqslant c_{13}(T). \tag{6.23}$$

We introduce the Banach space

$$\mathbb{Q} \equiv \bigl\{v(t) \mid v \in \mathbb{L}^2(0,T;\mathbb{B}),\ v' \in \mathbb{L}^2(0,T;\overline{\mathbb{W}}^*)\bigr\}.$$

Assuming that there is a completely continuous embedding $\mathbb{B} \hookrightarrow\hookrightarrow \mathbb{V}^* \subset \overline{\mathbb{W}}^*$, we obtain from the well-known Lyons–Aubin compactness theorem that there is a completely continuous embedding $\mathbb{Q} \hookrightarrow\hookrightarrow \mathbb{L}^2(0,T;\mathbb{V}^*)$.

It follows from the *a priori* estimates (6.21) and (6.23) that the sequence $\{\mathbb{A}u_m'\}$ is uniformly bounded with respect to $m \in \mathbb{N}$ in the Banach space $\mathbb{Q}$. Since the operator $\mathbb{A}\colon \mathbb{V} \to \mathbb{V}^*$ is bounded, we get the following *a priori* estimate from (6.11):

$$\|\mathbb{A}u_m'\|_* \leqslant c_{14}(T). \tag{6.24}$$

Step 4. *Passage to the limit.* The resulting *a priori* estimates (6.11)–(6.13), (6.21) and (6.23) enable us to conclude that the sequence $\{u_m\}$ has a subsequence with the following limit properties:

$$u_m \overset{*}{\rightharpoonup} u \quad *\text{-weakly in } \mathbb{L}^\infty(0,T;\mathbb{W}_1), \tag{6.25}$$

$$u_m' \overset{*}{\rightharpoonup} u' \quad *\text{-weakly in } \mathbb{L}^\infty(0,T;\mathbb{V}), \tag{6.26}$$

$$u_m' \rightharpoonup u' \quad \text{weakly in } \mathbb{L}^2(0,T;\mathbb{V}_0), \tag{6.27}$$

$$\mathbb{H}_f'(u_m) \overset{*}{\rightharpoonup} \chi(t) \quad *\text{-weakly in } \mathbb{L}^\infty(0,T;\mathbb{W}_1^*), \tag{6.28}$$

$$\mathbb{A}u_m' \to \mathbb{A}u' \quad \text{strongly in } \mathbb{L}^2(0,T;\mathbb{V}^*). \tag{6.29}$$

Since the linear operator $\mathbb{A}$ is coercive, it follows from the limit property (6.29) that

$$u'_m \to u' \quad \text{strongly in } \ \mathbb{L}^2(0,T;\mathbb{V}). \tag{6.30}$$

We now consider all terms subordinate to $\mathbb{H}'_f(u_m)$, starting with $\mathbb{F}'_f(u_m)$. We require that there are completely continuous and continuous embeddings $\mathbb{W}_1 \hookrightarrow\hookrightarrow \mathbb{W}_2 \subset \mathbb{V}$. We introduce the Banach space

$$\mathbb{Q}_1 \equiv \bigl\{v(t) \mid v \in \mathbb{L}^r(0,T;\mathbb{W}_1),\ v' \in \mathbb{L}^2(0,T;\mathbb{V})\bigr\}, \qquad r > 1.$$

By the Lyons–Aubin theorem we have a completely continuous embedding

$$\mathbb{Q}_1 \hookrightarrow\hookrightarrow \mathbb{L}^r(0,T;\mathbb{W}_2).$$

It follows from the *a priori* estimate (6.11) that the sequence $\{u_m\}$ is uniformly bounded with respect to $m \in \mathbb{N}$ in the Banach space $\mathbb{Q}_1$. Hence some subsequence of $\{u_m\}$ has the limit property

$$u_m \to u \quad \text{strongly in } \mathbb{L}^r(0,T;\mathbb{W}_2) \text{ for every fixed } r > 1. \tag{6.31}$$

Since the operator $\mathbb{F}'_f(v)$ is boundedly Lipschitz continuous, we arrive at the limit equation

$$\mathbb{F}'_f(u_m) \to \mathbb{F}'_f(u) \quad \text{strongly in } \ \mathbb{L}^r(0,T;\mathbb{W}_2^*), \quad r > 1. \tag{6.32}$$

We now consider the terms $\mathbb{A}_0 u'_m$, $\mathbb{A}'_{jf}(u_m)u'_m$. To treat them, we require that there are completely continuous and continuous embeddings

$$\mathbb{W}_1 \hookrightarrow\hookrightarrow \mathbb{V}_j \subset \mathbb{V}, \qquad j = 0,\ldots,n.$$

As above, by the Lyons–Aubin compactness theorem we have a completely continuous embedding

$$\mathbb{Q}_1 \hookrightarrow\hookrightarrow \mathbb{L}^r(0,T;\mathbb{V}_j), \qquad j = 0,\ldots,n, \quad r > 1. \tag{6.33}$$

Hence the sequence $\{u_m\}$ has a subsequence with the limit property

$$u_m \to u \quad \text{strongly in } \ \mathbb{L}^r(0,T;\mathbb{V}_j), \quad j = 0,\ldots,n. \tag{6.34}$$

By the properties of the operators $\mathbb{A}_j$ for $j = 1,\ldots,n$ we have

$$\mathbb{A}'_{jf}(u_m) \to \mathbb{A}'_{jf}(u) \quad \text{strongly in } \ \mathbb{L}^r(0,T;\mathcal{L}(\mathbb{V}_j,\mathbb{V}_j^*)). \tag{6.35}$$

It now follows from the properties of $\mathbb{A}_j$ that

$$\mathbb{A}_j(u_m) \to \mathbb{A}_j(u) \quad \text{strongly in } \ \mathbb{L}^r(0,T;\mathbb{V}_j^*), \quad r > 1.$$

On one hand, this yields that

$$\frac{d}{dt}\mathbb{A}_j(u_m) \overset{*}{\rightharpoonup} \frac{d}{dt}\mathbb{A}_j(u) \quad *\text{-weakly in } \ \mathcal{D}'(0,T;\mathbb{V}_j^*). \tag{6.36}$$

On the other hand, it follows from the *a priori* estimate (6.18) that

$$\frac{d}{dt}\mathbb{A}_j(u_m) \rightharpoonup \varphi(t) \quad \text{weakly in } \mathbb{L}^2(0,T;\mathbb{V}_j^*), \quad j=0,\dots,n. \tag{6.37}$$

Comparing (6.36) and (6.37) and taking into account that the weak* topology on $\mathcal{D}'(0,T;\mathbb{V}_j^*)$ is separable, we conclude that

$$\frac{d}{dt}\mathbb{A}_j(u_m) \rightharpoonup \frac{d}{dt}\mathbb{A}_j(u) \quad \text{weakly in } \mathbb{L}^2(0,T;\mathbb{V}_j^*), \quad j=0,\dots,n. \tag{6.38}$$

Regarding both sides of (6.1) as elements of $\mathcal{D}'(0,T)$, we act by them on an arbitrary function $\psi(t)\in\mathcal{D}(0,T)$ and obtain the equality

$$\begin{aligned}\Big\langle\Big\langle \frac{d}{dt}\langle \mathbb{A}u_m', w_k\rangle + \sum_{j=0}^{n}\Big\langle \frac{d}{dt}\mathbb{A}_j(u_m), w_k\Big\rangle_j \\ + (\mathbb{H}_f'(u_m), w_k)_1 - (\mathbb{F}_f'(u_m), w_k)_2,\ \psi(t)\Big\rangle\Big\rangle_{\mathcal{D}} = 0,\end{aligned} \tag{6.39}$$

where $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle_{\mathcal{D}}$ are the duality brackets between $\mathcal{D}(0,T)$ and $\mathcal{D}'(0,T)$. Using the limit properties obtained above, we can now pass to the limit as $m\to+\infty$ in (6.39) and obtain

$$\begin{aligned}\Big\langle\Big\langle \frac{d}{dt}\langle \mathbb{A}u', w\rangle + \sum_{j=0}^{n}\Big\langle \frac{d}{dt}\mathbb{A}_j(u), w\Big\rangle_j \\ + (\chi(t), w)_1 - (\mathbb{F}_f'(u), w)_2,\ \psi(t)\Big\rangle\Big\rangle_{\mathcal{D}} = 0\end{aligned} \tag{6.40}$$

for all $w\in\mathbb{W}_1$ and all $\psi(t)\in\mathcal{D}(0,T)$. We want to prove that in fact $\chi(t)=\mathbb{H}_f'(u)$.

Step 5. *The monotonicity method.* To use the monotonicity method, we require that

$$(\mathbb{H}_f'(v_1)-\mathbb{H}_f'(v_2), v_1-v_2)_1 \geqslant 0 \tag{6.41}$$

for all $v_1, v_2\in\mathbb{W}_1$. Hence for all $v\in\mathbb{L}^p(0,T;\mathbb{W}_1)$ we have

$$0\leqslant X_m = \int_0^T (\mathbb{H}_f'(u_m)-\mathbb{H}_f'(v), u_m - v)_1\,dt. \tag{6.42}$$

The following equality holds:

$$\begin{aligned}\int_0^T (\mathbb{H}_f'(u_m), u_m)_1\,dt &= \int_0^T (\mathbb{F}_f'(u_m), u_m)_2\,dt \\ &- \int_0^T \langle \mathbb{A}u_m'', u_m\rangle\,dt + \frac12\langle \mathbb{A}_0 u_{m0}, u_{m0}\rangle_0 + \sum_{j=1}^{n}\frac{p_j-1}{p_j}\langle \mathbb{A}_j(u_{m0}), u_{m0}\rangle_j \\ &- \frac12\langle \mathbb{A}_0 u_{mT}, u_{mT}\rangle_0 - \sum_{j=1}^{n}\frac{p_j-1}{p_j}\langle \mathbb{A}_j(u_{mT}), u_{mT}\rangle_j,\end{aligned} \tag{6.43}$$

where $u_{mT} = u_m(T)$. Note also that

$$-\int_0^T \langle \mathbb{A}u''_m, u_m\rangle\,dt = -\int_0^T \frac{d}{dt}\langle \mathbb{A}u'_m, u_m\rangle\,dt + \int_0^T \langle \mathbb{A}u'_m, u'_m\rangle\,dt$$
$$= \langle \mathbb{A}u_{m0}, u_{m1}\rangle - \langle \mathbb{A}u_{mT}, u'_{mT}\rangle + \int_0^T \langle \mathbb{A}u'_m, u'_m\rangle\,dt. \quad (6.44)$$

By what was proved above, the sequence $\{u_m(T)\}$ is bounded in $\mathbb{W}_1$ uniformly with respect to $m \in \mathbb{N}$, and the sequence $\{u'_m(T)\}$ is bounded in $\mathbb{V}$ uniformly with respect to $m \in \mathbb{N}$. Since the embedding $\mathbb{W}_1 \hookrightarrow\hookrightarrow \mathbb{V}$ is completely continuous, there is a subsequence of $\{u_m\}$ such that

$$u_m(T) \to \xi_1 \quad \text{strongly in } \ \mathbb{V}, \quad (6.45)$$
$$u'_m(T) \rightharpoonup \xi_2 \quad \text{weakly in } \ \mathbb{V}. \quad (6.46)$$

Since we have proved that

$$u(t) \in \mathbb{L}^\infty(0,T;\mathbb{W}_1), \quad u'(t) \in \mathbb{L}^2(0,T;\mathbb{V}_0), \quad \frac{d}{dt}\mathbb{A}u'(t) \in \mathbb{L}^2(0,T;\overline{\mathbb{W}}^*),$$

it follows that, after a possible alteration of $u(t)$ on a subset of Lebesgue measure 0 in $[0,T]$, we have

$$u(t) \in \mathbb{C}([0,T];\mathbb{V}_0), \qquad \mathbb{A}u'(t) \in \mathbb{C}([0,T];\overline{\mathbb{W}}^*).$$

Therefore $u(0)$, $u(T)$, $\mathbb{A}u'(0)$ and $\mathbb{A}u'(T)$ are well defined. By what was proved above,

$$u'_m \to u' \quad \text{strongly in } \ \mathbb{L}^2(0,T;\mathbb{V})$$

and, therefore, for a subsequence of $\{u_m\}$ we have

$$u'_m(t) \to u'(t) \quad \text{strongly in } \mathbb{V} \text{ for almost all } t \in [0,T].$$

Therefore $\xi_2 = u'(T)$ if we slightly decrease $T > 0$. Moreover, the following limit property holds by (6.34):

$$u_m \to u \quad \text{strongly in } \ \mathbb{L}^2(0,T;\mathbb{V}_0).$$

Therefore for a subsequence we have

$$u_m(t) \to u(t) \quad \text{strongly in } \mathbb{V}_0 \subset \mathbb{V} \text{ for almost all } t \in [0,T].$$

Hence $\xi_1 = u(T)$ if we slightly decrease $T > 0$. Using the limiting relations obtained above and the continuity of the linear operator $\mathbb{A}$, we arrive at the following limit property for a certain subsequence (which we regard for the moment as finalized) of $\{u_m\}$:

$$-\lim_{m\to+\infty}\int_0^T \langle \mathbb{A}u''_m, u_m\rangle\,dt = \langle \mathbb{A}u_0, u_1\rangle - \langle \mathbb{A}u(T), u'(T)\rangle + \int_0^T \langle \mathbb{A}u', u'\rangle\,dt. \quad (6.47)$$

By our overall conditions, the expression $\langle \mathbb{A}_j(v), v\rangle_j^{1/p_j}$, $j = 0, \ldots, n$, is a norm on the Banach space $\mathbb{V}_j$ inducing its original topology. Combining this with the *a priori* estimate $|u_m|_1 \leqslant c_4(T)$ and the completely continuous embedding $\mathbb{W}_1 \hookrightarrow\hookrightarrow \mathbb{V}_j$, we have, as above,

$$u_m(T) \to u(T) \quad \text{strongly in } \ \mathbb{V}_j. \tag{6.48}$$

Moreover, for some subsequence of $\{u_m\}$ we have

$$\lim_{m\to+\infty} \langle \mathbb{A}_j(u_m(T)), u_m(T)\rangle_j^{1/p_j} = \langle \mathbb{A}_j(u(T)), u(T)\rangle_j^{1/p_j}, \qquad j = 0, \ldots, n.$$

On the other hand, the following limit property holds because of the initial conditions (6.2) and (6.3):

$$\lim_{m\to+\infty} \langle \mathbb{A}_j(u_{m0}), u_{m0}\rangle_j^{1/p_j} = \langle \mathbb{A}_j(u_0), u_0\rangle_j^{1/p_j}, \qquad j = 0, \ldots, n.$$

Combining all these limit properties, we finally get

$$\begin{aligned}\lim_{m\to+\infty} \int_0^T (\mathbb{H}'_f(u_m), u_m)_1\, dt &= \int_0^T (\mathbb{F}'_f(u), u)_2\, dt \\ &+ \langle \mathbb{A}u_0, u_1\rangle - \langle \mathbb{A}u(T), u'(T)\rangle + \int_0^T \langle \mathbb{A}u', u'\rangle\, dt + \frac{1}{2}\langle \mathbb{A}_0 u_0, u_0\rangle_0 \\ &+ \sum_{j=1}^n \frac{p_j - 1}{p_j}\langle \mathbb{A}_j(u_0), u_0\rangle_j - \frac{1}{2}\langle \mathbb{A}_0 u(T), u(T)\rangle_0 - \sum_{j=1}^n \frac{p_j-1}{p_j}\langle \mathbb{A}_j(u(T)), u(T)\rangle_j.\end{aligned} \tag{6.49}$$

We now let $m \to +\infty$ in (6.42) and obtain that

$$\begin{aligned}0 \leqslant &- \int_0^T (\mathbb{H}'_f(v), u - v)_1\, dt - \int_0^T (\chi, v)_1\, dt + \int_0^T (\mathbb{F}'_f(u), u)_2\, dt \\ &+ \langle \mathbb{A}u_0, u_1\rangle - \langle \mathbb{A}u(T), u'(T)\rangle + \int_0^T \langle \mathbb{A}u', u'\rangle\, dt + \frac{1}{2}\langle \mathbb{A}_0 u_0, u_0\rangle_0 \\ &+ \sum_{j=1}^n \frac{p_j - 1}{p_j}\langle \mathbb{A}_j(u_0), u_0\rangle_j - \frac{1}{2}\langle \mathbb{A}_0 u(T), u(T)\rangle_0 - \sum_{j=1}^n \frac{p_j-1}{p_j}\langle \mathbb{A}_j(u(T)), u(T)\rangle_j.\end{aligned} \tag{6.50}$$

We multiply both sides of (6.1) by a function $\varphi(t) \in \mathbb{C}^{(1)}([0,T])$ and integrate over $t \in [0,T]$. Note that we have

$$\begin{aligned}&\int_0^T \langle \mathbb{A}u''_m, \varphi(t)w\rangle\, dt \\ &\qquad = -\int_0^T \langle \mathbb{A}u'_m, \varphi' w\rangle\, dt - \langle \mathbb{A}u_{1m}, \varphi(0)w\rangle + \langle \mathbb{A}u'_{Tm}, \varphi(T)w\rangle\end{aligned} \tag{6.51}$$

for all $w \in \overline{\mathbb{W}}_m$ (we recall that $\overline{\mathbb{W}}_m$ is the linear span of the functions $w_1, \dots$ $\dots, w_m$). Integrating by parts in (6.51) and passing to the limit, we obtain that

$$\begin{aligned} -\int_0^T \langle \mathbb{A}u', \varphi' w\rangle\, dt - \langle \mathbb{A}u_1, \varphi(0)w\rangle + \langle \mathbb{A}u'(T), \varphi(T)w\rangle \\ + \int_0^T \sum_{j=0}^n \left\langle \frac{d}{dt}\mathbb{A}_j(u), \varphi(t)w\right\rangle_j dt + \int_0^T (\chi(t), \varphi(t)w)_1\, dt \\ - \int_0^T (\mathbb{F}'_f(u), \varphi(t)w)_2\, dt = 0. \end{aligned} \tag{6.52}$$

We introduce the space

$$D \equiv \{v(t) \mid v \in \mathbb{L}^2(0,T;\mathbb{W}_1),\ v' \in \mathbb{L}^2(0,T;\mathbb{V})\}, \tag{6.53}$$

which is a reflexive Banach space with norm

$$\|v\|_D = \left(\int_0^T (|v|_1^2 + \|v'\|^2)\, dt\right)^{1/2}.$$

Note that the sets of the form

$$\left\{\sum_{k=1}^m \varphi_k(t)w_k,\ \varphi_k(t) \in \mathbb{C}^{(1)}([0,T]),\ w_k \in \mathbb{W}_1\right\} \tag{6.54}$$

are dense in D. Indeed, the following theorem holds.

Theorem 6.4. *Let L_1, L_2 be Banach spaces with a dense continuous embedding $L_1 \overset{\mathrm{ds}}{\subset} L_2$. We define a Banach space D as the set*

$$\{v \mid v \in \mathbb{L}^2(0,T;L_1),\ v' \in \mathbb{L}^2(0,T;L_2)\}$$

with norm

$$\|v\|_D^2 = \|v\|^2_{\mathbb{L}^2(0,T;L_1)} + \|v'\|^2_{\mathbb{L}^2(0,T;L_2)}.$$

Then the set $\mathcal{C}^1$ of elements of the form $\sum_{k=1}^m \varphi_k(t)w_k$, where $w_k \in L_1$ and $\varphi_k \in \mathbb{C}^{(1)}([0,T])$, is dense in D.

Proof. Clearly, it suffices to show that for every element of D one can find an element of $\mathcal{C}^1$ arbitrarily close to it in the norm of D. This will be done in several steps.

1. We first prove the following lemma.

Lemma 6.5. *The set $\mathbb{C}([0,T];L_2)$ is dense in $\mathbb{L}^2(0,T;L_2)$.*

Proof. It is known [15] that the piecewise-constant functions are dense in $\mathbb{L}^2(0,T;L_2)$. Furthermore, linear interpolation in a sufficiently small neighbourhood of the boundary of the set of discontinuity enables us to approximate every piecewise-constant function within any desired accuracy in $\mathbb{L}^2(0,T;L_2)$ by a continuous function. Indeed, let w_1 (resp. w_2) be the value of a piecewise-constant function $w(t)$ in some left (resp. right) half-neighbourhood of t_0. The corresponding linear interpolant $w_0(t)$ on the interval $[t_0-\delta, t_0+\delta]$ is given by

$$w_0(t) = w_1 + \frac{t-(t_0-\delta)}{2\delta}(w_2-w_1) \equiv w_2 - \frac{(t_0+\delta)-t}{2\delta}(w_1-w_2).$$

Then we have

$$\begin{aligned}&\int_{t_0-\delta}^{t_0+\delta} \|w_0(t)-w(0)\|_{L_2}^2\,dt\\ &\quad= \int_{t_0-\delta}^{t_0} \left\|w_1 + \frac{t-(t_0-\delta)}{2\delta}(w_2-w_1) - w_1\right\|_{L_2}^2 dt\\ &\qquad+ \int_{t_0}^{t_0+\delta} \left\|w_2 - \frac{t_0+\delta-t}{2\delta}(w_1-w_2) - w_2\right\|_{L_2}^2 dt \leqslant \frac{\delta^2}{3}\|w_2-w_1\|_{L_2}^2.\end{aligned}$$

Summing such terms over all points of discontinuity (there are finitely many of them) and choosing a sufficiently small δ, we get the desired result. $\square$

2. Let v be an arbitrary element of D. We put $w = v'$. Then $w \in \mathbb{L}^2(0,T;L_2)$. For brevity we put

$$L_3 = \mathbb{L}^2(0,T;L_1), \qquad L_4 = \mathbb{L}^2(0,T;L_2).$$

We can write

$$v(t) = v(0) + \int_0^t w(\tau)\,d\tau.$$

3. In view of Lemma 6.5, for every $\varepsilon > 0$ there is a $w_1(t) \in \mathbb{C}([0,T];L_2)$ such that

$$\|w(t)-w_1(t)\|_{L_4} < \varepsilon.$$

It is known [15] that w_1 can be approximated arbitrarily closely in the norm of $\mathbb{C}([0,T];L_2)$ (and, therefore, in the norm of $L_4 \equiv \mathbb{L}^2(0,T;L_2)$) by a polynomial with coefficients in L_2, that is, by an element of the form

$$P_1(t) = \sum_{j=0}^{k} a_j t^j, \qquad a_j \in L_2.$$

4. Since the embedding $L_1 \overset{\mathrm{ds}}{\subset} L_2$ is dense, the coefficients of $P_1(t)$ (we recall that they belong to L_2) can be approximated arbitrarily closely by elements of L_1. The resulting polynomial $P_2(t)$ will be arbitrarily close to $P_1(t)$ in the norm of $\mathbb{C}([0,T];L_2)$ and, therefore, in the norm of L_4. The same dense embedding enables us to find an element $v_{01} \in L_1$ arbitrarily close to $v(0)$. We put

$$v_1(t) = v(t) + (v_{01} - v_1).$$

Clearly,

$$v_1'(t) = v'(t) = w(t).$$

5. We define the element

$$v_2(t) = v_{01} + \int_0^t P_2(\tau)\,d\tau.$$

Our previous argument enables us to make the function $w_2'(t) = P_2(t)$ arbitrarily close to $v'(t) = w(t)$ in the norm of L_4. We also have

$$\begin{aligned}\|v_1(t) - v_2(t)\|_{L_3}^2 &= \int_0^T \|v_1(t) - v_2(t)\|_{L_1}^2\,dt = \int_0^T \left\| \int_0^t (w(\tau) - P_2(\tau))\,d\tau \right\|_{L_1}^2 dt \\ &\leqslant \int_0^T \left(\int_0^t \|w(\tau) - P_2(\tau)\|_{L_2}\,d\tau \right)^2 dt = \int_0^T \left(\int_0^t 1 \cdot \|w(\tau) - P_2(\tau)\|_{L_2}\,d\tau \right)^2 dt \\ &\leqslant \int_0^T T \int_0^t \|w(\tau) - P_2(\tau)\|_{L_2}^2\,d\tau\,dt \leqslant T^2 \|w(t) - P_2(t)\|_{L_4}^2.\end{aligned}$$

This bound shows that $\|v_1(t) - v_2(t)\|_{L_3}$ can be made arbitrarily small provided that $\|w(t) - P_2(t)\|_{L_4}$ is sufficiently small. Furthermore,

$$\|v(t) - v_1(t)\|_{L_3} \equiv \|v(0) - v_{01}\|_{L_1}$$

can also be made arbitrarily small by what was said above. By parts 1–4, for every $\varepsilon > 0$ one can choose $P_2(t)$ such that $\|w(t) - P_2(t)\|_{L_4} < \varepsilon$. Therefore we obtain that every element $v(t) \in D$ can be approximated within any given accuracy by an element

$$v_{01} + \int_0^t P_2(\tau)\,d\tau = v_{01} + \sum_{j=0}^{k} \frac{a_j}{j+1} t^{j+1}, \qquad v_{01}, a_j \in L_1.$$

This proves the theorem. □

By Step 4 (see (6.25), (6.30)) we have

$$u(t) \in \mathbb{L}^\infty(0,T;\mathbb{W}_1) \subset \mathbb{L}^2(0,T;\mathbb{W}_1), \qquad u'(t) \in \mathbb{L}^2(0,T;\mathbb{V}).$$

Then $u \in \mathbb{D}$. There is also a sequence

$$\{u_l\}, \qquad u_l = \sum_{k=1}^{l} \varphi_{kl}(t) w_k, \qquad \varphi_{kl} \in \mathbb{C}^{(1)}[0,T],$$

with the limit property

$$\int_0^T \left(|u - u_l|_1^2 + \|u' - u_l'\|^2 \right) dt \to +0 \qquad \text{as} \quad l \to +\infty. \tag{6.55}$$

We now prove an auxiliary assertion.

Lemma 6.6. *There is a subsequence of* $\{u_l\}$ *such that*

$$\lim_{l\to+\infty} \|u_l(t) - u(t)\| = 0 \quad \text{uniformly with respect to} \ \ t \in [0,T]. \tag{6.56}$$

Proof. We use the technique in [16]. By what was proved at Step 4 (see (6.25), (6.30)) we have $u \in \mathbb{L}^\infty(0,T;\mathbb{W}_1)$, $u' \in \mathbb{L}^2(0,T;\mathbb{V})$. Then the following chain of inequalities holds:

$$\|u(t_1) - u(t_2)\| \leqslant \int_{t_1}^{t_2} \|u'\|\,dt \leqslant \left(\int_{t_1}^{t_2} \|u'\|^2\,dt\right)^{1/2} \sqrt{|t_1 - t_2|} \leqslant c(T)\sqrt{|t_1 - t_2|}. \tag{6.57}$$

Hence the function $u(t)$ is uniformly continuous on $[0,T]$ with values in $\mathbb{V}$.

We note from (6.55) that the sequence $\{u_l\}$ is uniformly bounded in $\mathbb{L}^2(0,T;\mathbb{W}_1)$ while the sequence $\{u_l'\}$ is uniformly bounded in $\mathbb{L}^2(0,T;\mathbb{V})$. Since $\mathbb{W}_1$ is compactly embedded in $\mathbb{V}$, the Lyons–Aubin theorem enables us to find a subsequence of $\{u_l\}$ such that

$$u_l(t) \to u(t) \quad \text{strongly in} \quad \mathbb{L}^2(0,T;\mathbb{V}),$$
$$u_l(t) \to u(t) \quad \text{strongly in} \quad \mathbb{V}$$

for almost all $t \in [0,T]$. Hence there is a countable everywhere-dense set $E = \{t_k\}_{k=1}^\infty \subset [0,T]$ such that $u_l(t) \to u(t)$ strongly in $\mathbb{V}$ on E. We claim that then

$$u_l(t) \to u(t) \quad \text{strongly in} \quad \mathbb{V}$$

uniformly on $[0,T]$. Indeed, we first of all deduce from the boundedness of the sequence $\{u_l'\}$ in $\mathbb{L}^2(0,T;\mathbb{V})$ (see above) and the estimate

$$\|u_l(t) - u_l(t^*)\| \leqslant \int_{t^*}^{t} \|u_l'\|\,ds \leqslant \left(\int_{t^*}^{t} \|u_l'\|^2\,ds\right)^{1/2} \sqrt{|t - t^*|} \leqslant c(T)\sqrt{|t - t^*|} \tag{6.58}$$

that $\{u_l\}$ is an equicontinuous sequence of functions on $[0,T]$ with values in $\mathbb{V}$.

Given any $\varepsilon > 0$, we shall now construct an $L(\varepsilon)$ such that for all $l > L(\varepsilon)$ we have

$$\|u_l(t) - u(t)\| < \varepsilon \quad \forall\, t \in [0,T]. \tag{6.59}$$

This is nothing other than the desired uniform convergence. Using the equicontinuity of $\{u_l\}$ and (6.57), we choose $\delta = \delta(\varepsilon/3)$ such that for $|\bar{t} - \bar{\bar{t}}| < \delta$ we have

$$\|u_l(\bar{t}) - u_l(\bar{\bar{t}})\| < \frac{\varepsilon}{3}, \qquad \|u(\bar{t}) - u(\bar{\bar{t}})\| < \frac{\varepsilon}{3}. \tag{6.60}$$

Then we choose $K = K(\delta)$ in such a way that all pairwise distances between the points $\{t_k\}_{k=1}^K \subset E$ are less than δ (this is possible since E is dense on $[0,T]$). We further choose $L = L(\varepsilon/3)$ in such a way that

$$\|u_l(t_k) - u(t_k)\| < \frac{\varepsilon}{3}$$

for all t_k, $k = 1, \dots, K$ (finitely many points), and all $l > L$ (this is possible because $u_l(t) \to u(t)$ in $\mathbb{V}$ on E). This is the desired L. Indeed, consider the inequality

$$\|u_l(t) - u(t)\| \leqslant \|u_l(t) - u_l(t_k)\| + \|u_l(t_k) - u(t_k)\| + \|u(t_k) - u(t)\|,$$

where $t \in [0, T]$, $l > L$, and the point t_k is chosen using the conditions $|t - t_k| < \delta$, $k \in \{1, \ldots, K\}$ (this is possible by the construction of K). Then, by the choice of δ,

$$\|u_l(t) - u_l(t_k)\| < \frac{\varepsilon}{3}, \qquad \|u(t) - u(t_k)\| < \frac{\varepsilon}{3}.$$

By the choice of L we get

$$\|u_l(t_k) - u(t_k)\| < \frac{\varepsilon}{3}.$$

This proves (6.59). □

Using Lemma 6.6, we conclude that there is a subsequence of $\{u_l\}$ possessing the following limit properties in addition to (6.55):

$$\|u(0) - u_l(0)\| \to +0, \qquad l \to +\infty, \tag{6.61}$$

$$\|u(T) - u_l(T)\| \to +0, \qquad l \to +\infty. \tag{6.62}$$

Since the duality brackets are linear with respect to their second arguments, one can rewrite (6.52) in the form

$$-\int_0^T \langle \mathbb{A}u', u_l' \rangle \, dt - \langle \mathbb{A}u_1, u_l(0) \rangle + \langle \mathbb{A}u'(T), u_l(T) \rangle \\ + \int_0^T \sum_{j=0}^n \left\langle \frac{d}{dt} \mathbb{A}_j(u), u_l \right\rangle_j dt + \int_0^T (\chi(t), u_l)_1 \, dt - \int_0^T (\mathbb{F}_f'(u), u_l)_2 \, dt = 0. \tag{6.63}$$

Using the limit properties (6.55), (6.61) and (6.62), we can pass to the limit as $l \to +\infty$ in (6.63) and obtain

$$-\int_0^T \langle \mathbb{A}u', u' \rangle \, dt - \langle \mathbb{A}u_1, u_0 \rangle + \langle \mathbb{A}u'(T), u(T) \rangle \\ + \int_0^T \sum_{j=0}^n \left\langle \frac{d}{dt} \mathbb{A}_j(u), u \right\rangle_j dt + \int_0^T (\chi(t), u)_1 \, dt - \int_0^T (\mathbb{F}_f'(u), u)_2 \, dt = 0. \tag{6.64}$$

We now deal with the terms

$$\int_0^T \left\langle \frac{d}{dt} \mathbb{A}_j(u), u \right\rangle_j dt, \qquad j = 0, \ldots, n.$$

To do this, we deduce from the property

$$\|\mathbb{A}_{jf}'(u)\|_{\mathbb{V}_j \to \mathbb{V}_j^*} \leqslant \overline{M}_j \|u\|_j^{p_j - 2}$$

that

$$\|\mathbb{A}_j(v_1) - \mathbb{A}_j(v_2)\|_j^* \leqslant \mu_j(R) \|v_1 - v_2\|_j, \tag{6.65}$$

where

$$\mu_j(R) = c_j R, \qquad R = \max\{\|v_1\|_j^{p_j - 2}, \|v_2\|_j^{p_j - 2}\}.$$

Hence there is a chain of inequalities

$$\begin{aligned}\int_0^T (\|\mathbb{A}_j(v_1)-\mathbb{A}_j(v_2)\|_j^*)^{p_j'}\,dt \\ &\leqslant c_j^{p_j'}\int_0^T \max\bigl\{\|v_1\|_j^{(p_j-2)p_j/(p_j-1)},\|v_2\|_j^{(p_j-2)p_j/(p_j-1)}\bigr\}\|v_1-v_2\|_j^{p_j/(p_j-1)}\,dt \\ &\leqslant c_j^{p_j'}\biggl(\int_0^T \max\bigl\{\|v_1\|_j^{p_j},\|v_2\|_j^{p_j}\bigr\}\,dt\biggr)^{(p_j-2)/(p_j-1)} \\ &\qquad\times\biggl(\int_0^T \|v_1-v_2\|_j^{p_j}\,dt\biggr)^{1/(p_j-1)}, \qquad p_j'=\frac{p_j}{p_j-1}. \end{aligned} \tag{6.66}$$

Consider the resulting sequence $\{u_m\}\subset\mathbb{C}^{(1)}([0,T];\mathbb{V}_j)$ of Galerkin approximants (we regard it as finalized for the moment). By the limit property (6.34),

$$\int_0^T \|u-u_m\|_j^{p_j}\,dt\to+0,\qquad m\to+\infty, \tag{6.67}$$

but then (6.66) yields that

$$\int_0^T (\|\mathbb{A}_j(u)-\mathbb{A}_j(u_m)\|_j^*)^{p_j'}\,dt\to+0,\qquad m\to+\infty. \tag{6.68}$$

We note that the following equality holds by Lemma 4.3:

$$\begin{aligned}\int_0^T \biggl\langle\frac{d}{dt}\mathbb{A}_j(u_m),u_m\biggr\rangle_j\,dt &= \frac{p_j-1}{p_j}\int_0^T \frac{d}{dt}\langle\mathbb{A}_j(u_m),u_m\rangle_j\,dt \\ &=\frac{p_j-1}{p_j}\bigl[\langle\mathbb{A}_j(u_m)(T),u_m(T)\rangle_j-\langle\mathbb{A}_j(u_m(0)),u_m(0)\rangle_j\bigr],\qquad j=0,\dots,n.\end{aligned} \tag{6.69}$$

Moreover, by the initial condition (6.2) and the limit property (6.48) we have

$$\lim_{m\to+\infty}\bigl\langle\mathbb{A}_j(u_m)(T),u_m(T)\bigr\rangle_j=\langle\mathbb{A}_j(u)(T),u(T)\rangle_j, \tag{6.70}$$

$$\lim_{m\to+\infty}\bigl\langle\mathbb{A}_j(u_m)(0),u_m(0)\bigr\rangle_j=\langle\mathbb{A}_j(u_0),u_0\rangle_j. \tag{6.71}$$

We note that if a sequence converges weakly in $\mathbb{L}^2(0,T;\mathbb{V}_j^*)$, then it also converges weakly in $\mathbb{L}^{p_j'}(0,T;\mathbb{V}_j^*)$ for $p_j\geqslant 2$ (since $p_j'\leqslant 2$). Using this and the limit property (6.38), we finally conclude that there is a subsequence of $\{u_m\}$ such that

$$\frac{d}{dt}\mathbb{A}_j(u_m)\rightharpoonup\frac{d}{dt}\mathbb{A}_j(u)\quad\text{weakly in }\mathbb{L}^{p_j'}(0,T;\mathbb{V}_j^*),\qquad j=0,\dots,n. \tag{6.72}$$

Therefore, using the limit properties (6.67) and (6.70)–(6.72), we can pass to the limit as $m\to+\infty$ in the subsequence $\{u_m\}$ (finalized at the moment) and obtain

$$\begin{aligned}\int_0^T \biggl\langle\frac{d}{dt}\mathbb{A}_j(u),u\biggr\rangle_j\,dt \\ &=\frac{p_j-1}{p_j}\bigl[\langle\mathbb{A}_j(u(T)),u(T)\rangle_j-\langle\mathbb{A}_j(u_0),u_0\rangle_j\bigr],\qquad j=0,\dots,n.\end{aligned} \tag{6.73}$$

Using this and (6.64), we arrive at the equality

$$
\begin{aligned}
-\int_0^T \langle \mathbb{A}u', u' \rangle \, dt - \langle \mathbb{A}u_1, u_0 \rangle + \langle \mathbb{A}u'(T), u(T) \rangle \\
+ \sum_{j=0}^{n} \frac{p_j - 1}{p_j} \bigl[\langle \mathbb{A}_j(u(T)), u(T) \rangle_j - \langle \mathbb{A}_j(u_0), u_0 \rangle_j \bigr] \\
+ \int_0^T (\chi(t), u)_1 \, dt - \int_0^T (\mathbb{F}'_f(u), u)_2 \, dt = 0.
\end{aligned} \tag{6.74}
$$

This and (6.50) in their turn yield the inequality

$$
0 \leqslant \int_0^T (\chi - \mathbb{H}'_f(v), u - v)_1 \, dt, \qquad v \in \mathbb{L}^p(0, T; \mathbb{W}_1). \tag{6.75}
$$

The following argument is standard in the monotonicity method, but we give it for completeness. Put $v = u - \lambda w$, where $w \in \mathbb{L}^p(0, T; \mathbb{W}_1)$, $\lambda > 0$. Then

$$
0 \leqslant \int_0^T (\chi - \mathbb{H}'_f(u - \lambda w), w)_1 \, dt.
$$

Letting $\lambda \to +0$ (this is possible since the operator $H'_f(\,\cdot\,)$ is boundedly Lipschitz continuous), we get the inequality

$$
0 \leqslant \int_0^T (\chi - \mathbb{H}'_f(u), w)_1 \, dt, \qquad w \in \mathbb{L}^p(0, T; \mathbb{W}_1).
$$

This is a contradiction if $\chi(t) \neq \mathbb{H}'_f(u)$.

Thus the local solubility is proved. In what follows we need another result on the strong convergence of the sequence of Galerkin approximants. To state it, we require that the following additional condition holds:

$$
\bigl(\mathbb{H}'_f(v_1) - \mathbb{H}'_f(v_2), v_1 - v_2\bigr)_1 \geqslant a|v_1 - v_2|_1^p, \qquad v_1, v_2 \in \mathbb{W}_1. \tag{6.76}
$$

Since the sequence u_m is bounded in $\mathbb{W}_1$ (see (6.11)), we have (after choosing a subsequence)

$$
u_m \rightharpoonup u \ \text{ weakly in } \ \mathbb{L}^p(0, T; \mathbb{W}_1), \qquad \mathbb{H}'_f(u_m) \rightharpoonup \mathbb{H}'_f(u) \ \text{ weakly in } \ \mathbb{L}^{p'}(0, T; \mathbb{W}_1^*), \tag{6.77}
$$

and, moreover,

$$
\lim_{m \to +\infty} \int_0^T (\mathbb{H}'_f(u_m), u_m)_1 \, dt = \int_0^T (\mathbb{H}'_f(u), u)_1 \, dt. \tag{6.78}
$$

Thus we have the limit relation

$$
0 = \lim_{m \to +\infty} \int_0^T (\mathbb{H}'_f(u) - \mathbb{H}'_f(u_m), u - u_m)_1 \, dt \geqslant a \lim_{m \to +\infty} \int_0^T |u - u_m|_1^p \, dt. \tag{6.79}
$$

We conclude that

$$u_m \to u \quad \text{strongly in } \ \mathbb{L}^p(0,T;\mathbb{W}_1) \tag{6.80}$$

and, clearly (up to choosing a subsequence),

$$u_m \to u \quad \text{strongly in } \mathbb{W}_1 \text{ for almost all } t \in [0,T].$$

Step 6. *Continuation of solutions in time.* We now require the operator $\mathbb{A}$ to be coercive in the following weak sense:

$$\|\mathbb{A}w\|^*_{\overline{\mathbb{W}}} \geqslant d\|w\|^*_{\overline{\mathbb{W}}}, \qquad w \in \mathbb{V} \subset \overline{\mathbb{W}}^*. \tag{6.81}$$

Then it follows from (6.21) that $u'' \in \mathbb{L}^2(0,T;\overline{\mathbb{W}}^*)$.

We introduce two classes of functions with values in Banach spaces:

$$R_1 = \{v\colon v(t) \in \mathbb{W}_1,\ v'(t) \in \mathbb{V}\}, \qquad R_2 = \{v\colon v(t) \in \mathbb{V},\ v'(t) \in \overline{\mathbb{W}}^*\}.$$

By definition of a weak generalized solution $u(t)$ we have $u(t) \in \mathbb{L}^\infty(0,T;R_1)$. Changing $u(t)$, if necessary, on a set of Lebesgue measure 0 in $[0,T]$, we get

$$u(t) \in \mathbb{C}([0,T];R_2). \tag{6.82}$$

Clearly, $R_1 \subset R_2$. By (6.82), the trace of the function $u(t)$ exists at every point $t \in [0,T]$. We introduce two functions,

$$\psi_1(T) = \operatorname*{ess.sup}_{t\in[0,T]}\bigl(|u|_1(t) + \|u'\|(t)\bigr), \qquad \psi_2(T) = \sup_{t\in[0,T]}\bigl(\|u\|(t) + \|u'\|_{\overline{\mathbb{W}}^*}(t)\bigr). \tag{6.83}$$

Let $T_0 > 0$ be such that the weak generalized solution of the problem exists for all $T < T_0$. Then either $T_0 = +\infty$, or $T_0 < +\infty$ and, in the latter case,

$$\lim_{T\uparrow T_0} \psi_1(T) = +\infty. \tag{6.84}$$

Indeed, if (6.84) does not hold, then $T_0 < +\infty$ and

$$\lim_{T\uparrow T_0} \psi_1(T) < +\infty. \tag{6.85}$$

However,

$$\lim_{T\uparrow T_0} \psi_2(T) \leqslant b \lim_{T\uparrow T_0} \psi_1(T) < +\infty.$$

Hence,

$$\sup_{t\in[0,T_0]}\bigl(\|u\|(t) + \|u'\|_{\overline{\mathbb{W}}^*}(t)\bigr) < +\infty. \tag{6.86}$$

First of all we note that for almost every $t \in [0,T]$ we can pass to the limit in (6.10) and obtain that

$$\frac{\mathrm{m}}{2}\|u'\|^2 + c_1|u|_1^p + \int_0^t \langle \mathbb{A}_0 u', u'\rangle_0\, ds \leqslant E(0) + \frac{c_2}{2}\int_0^t \bigl(\|u'\|^2 + |u|_1^{2q+2}\bigr)\, ds. \tag{6.87}$$

Combining this with (6.85) and the coercivity of $\mathbb{A}_0$, we see that $u' \in \mathbb{L}^2(0,T_0;\mathbb{V}_0)$.

The *a priori* estimate (6.21) takes the form

$$\int_0^T \left(\left\| \frac{d}{dt} \mathbb{A} u'_m \right\|^*_{\overline{\mathbb{W}}} \right)^2 dt \leqslant c_{12}(T) < +\infty. \tag{6.88}$$

We see from this that the sequence $\{u_m\}$ has a subsequence with the limit property

$$\frac{d}{dt}\mathbb{A}u'_m \rightharpoonup \frac{d}{dt}\mathbb{A}u' \quad \text{weakly in} \quad \mathbb{L}^2(0,T;\overline{\mathbb{W}}^*).$$

Then, using (6.88) and the weak sequential semicontinuity of the norm in a reflexive Banach space, we get the estimate

$$\int_0^T \left(\left\| \frac{d}{dt} \mathbb{A} u' \right\|^*_{\overline{\mathbb{W}}} \right)^2 dt \leqslant \liminf_{m\to+\infty} \int_0^T \left(\left\| \frac{d}{dt} \mathbb{A} u'_m \right\|^*_{\overline{\mathbb{W}}} \right)^2 dt \leqslant c_{12}(T) < +\infty. \tag{6.89}$$

We shall prove that $c_{12}(T) < +\infty$ for all $T \in [0, T_0]$. To do this, we consider equation (6.15). The main difficulty in getting a uniform *a priori* estimate for the expression on the left-hand side stems from the summand

$$\mathbb{P}^{\mathrm{t}}_m \sum_{j=1}^n \frac{d}{dt}\mathbb{A}_j(u_m).$$

The other summands are much easier to study once this one is understood. Now, by (6.17) and (6.6) we have the *a priori* estimates

$$\begin{aligned}
\int_0^T \left\| \mathbb{P}^{\mathrm{t}}_m \frac{d}{dt}\mathbb{A}_j(u_m) \right\|^2_{\overline{\mathbb{W}}^*} dt &\leqslant \int_0^T \left\| \frac{d}{dt}\mathbb{A}_j(u_m) \right\|^2_{\overline{\mathbb{W}}^*} dt \leqslant d_j \int_0^T \left\| \frac{d}{dt}\mathbb{A}_j(u_m) \right\|^2_{\mathbb{V}_j^*} dt \\
&\leqslant c_j \operatorname*{ess.sup}_{s\in[0,T]} |u_m(s)|_1^{p_j-2} \int_0^T \langle \mathbb{A}'_{jf}(u_m)u'_m, u'_m\rangle_j \, dt \\
&\leqslant c_j \operatorname*{ess.sup}_{s\in[0,T]} |u_m(s)|_1^{p_j-2} \left[E_m(0) + \int_0^T (\mathbb{F}'_f(u_m), u'_m)_2 \, dt \right].
\end{aligned} \tag{6.90}$$

Hence we get the estimate

$$\begin{aligned}
&\int_0^T \left\| \frac{d}{dt}\mathbb{A}_j(u_m) \right\|^2_{\overline{\mathbb{W}}^*} dt \\
&\qquad \leqslant c_j \operatorname*{ess.sup}_{s\in[0,T]} |u_m(s)|_1^{p_j-2} \left[E_m(0) + \int_0^T (\mathbb{F}'_f(u_m), u'_m)_2 \, dt \right].
\end{aligned} \tag{6.91}$$

By (6.10) we also have

$$\operatorname*{ess.sup}_{s\in[0,T]} |u_m(s)|_1^{p_j-2} \leqslant d_j \left(E_m(0) + \frac{c_2}{2} \int_0^T \left(\|u'_m\|^2 + |u_m|_1^{2q+2} \right) dt \right)^{(p_j-2)/p}. \tag{6.92}$$

Therefore (6.91) and (6.92) yield the desired estimate

$$\int_0^T \left\| \frac{d}{dt}\mathbb{A}_j(u_m)\right\|^2_{\mathbb{W}^*} dt \leqslant d_j\left(E_m(0) + \frac{c_2}{2}\int_0^T \left(\|u_m'\|^2 + |u_m|_1^{2q+2}\right) dt\right)^{(p_j-2)/p} \times \left[E_m(0) + \int_0^T (\mathbb{F}_f'(u_m), u_m')_2\, dt\right] = I_{1jm}. \quad (6.93)$$

The other summands in (6.15) can be estimated in a similar way, and we finally get the estimate

$$\int_0^T \left(\left\|\frac{d}{dt}\mathbb{A}u_m'\right\|^*_{\overline{\mathbb{W}}}\right)^2 dt \leqslant \sum_{j=1}^n I_{1jm} + I_{2m} + I_{3m} + I_{4m}.$$

We now let $m \to +\infty$ in both sides of this inequality. In particular, for I_{1jm} we use (6.27) and the continuous embedding $\mathbb{V}_0 \subset \mathbb{W}_2$ (along with the other limit properties obtained in this section by taking the lower limit as $m \to +\infty$ in the definition of I_{1jm}) to conclude that

$$\begin{aligned}\varliminf_{m\to+\infty} I_{1jm} &= d_j\left(E(0) + \frac{c_2}{2}\int_0^T \left(\|u'\|^2 + |u|_1^{2q+2}\right) ds\right)^{(p_j-2)/p} \times \left[E(0) + \int_0^T (\mathbb{F}_f'(u), u')_2\, dt\right] \\ &\leqslant d_j\left(E(0) + \frac{c_2}{2}\int_0^T \left(\|u'\|^2 + |u|_1^{2q+2}\right) ds\right)^{(p_j-2)/p} \\ &\quad\times \left[E(0) + \left(\int_0^T |\mathbb{F}_f'(u)|_2^*\, dt\right)^{1/2}\left(\int_0^T \|u'\|_0^2\, dt\right)^{1/2}\right]. \end{aligned} \quad (6.94)$$

Since $u(t) \in \mathbb{L}^\infty(0, T_0; R_1)$ and (by what was proved above) we have $u'(t) \in \mathbb{L}^2(0, T_0; \mathbb{V}_0)$, we obtain from (6.94) that

$$\varliminf_{m\to+\infty} I_{1jm} \leqslant c(T_0) < +\infty \quad (6.95)$$

for all $T \in [0, T_0]$. We similarly consider all other terms in the operator equality (6.15). Thus we arrive at the uniform bound

$$\int_0^{T_0} \left(\left\|\frac{d}{dt}\mathbb{A}u'\right\|^*_{\overline{\mathbb{W}}}\right)^2 dt \leqslant c(T_0) < +\infty. \quad (6.96)$$

It follows that

$$\begin{aligned}&\|\mathbb{A}u'(t_1) - \mathbb{A}u'(t_2)\|^*_{\overline{\mathbb{W}}} \\ &\quad\leqslant \int_{t_1}^{t_2} \left\|\frac{d}{dt}\mathbb{A}u'\right\|^*_{\overline{\mathbb{W}}} dt \leqslant |t_1 - t_2|^{1/2}\left(\int_0^{T_0}\left(\left\|\frac{d}{dt}\mathbb{A}u'\right\|^*_{\overline{\mathbb{W}}}\right)^2 dt\right)^{1/2}\end{aligned} \quad (6.97)$$

for all $t_1, t_2 \in [0, T_0)$. Hence we see from (6.97) and (6.81) that $u'(t)$ is a uniformly continuous $\overline{\mathbb{W}}^*$-valued function on the half-open interval $[0, T_0)$ and, by the proof of Lemma 6.6, $u(t)$ is a uniformly continuous $\mathbb{V}$-valued function on $[0, T_0)$. Thus there is a continuous extension of $u(t)$ such that $u(T_0) \in R_2$. As a result, we obtain that

$$u(t) \in \mathbb{L}^\infty(0, T_0; R_1), \qquad u(t) \in \mathbb{C}([0, T_0]; R_2).$$

The second inclusion implies that there is an element $u(T_0) \in R_2$. We claim that $u(T_0) \in R_1$. Indeed, we put

$$E = \bigl\{t \in [0, T_0] \mid u(t) \in R_1,\ \|u(t)\|_{R_1} \leqslant \|u\|_{\mathbb{L}^\infty(0,T_0;R_1)}\bigr\}.$$

It follows from the first inclusion that $\overline{E} = [0, T_0]$. Consider an arbitrary sequence

$$t_n \to T_0 - 0, \qquad t_n \in E.$$

Then the sequence $\{u(t_n)\}$ is bounded in R_1 and, therefore, contains a subsequence $\{u(t_{n_k})\}$ weakly convergent to x in R_1. Since the embedding operator $R_1 \to R_2$ is continuous and every continuous operator is also continuous in the sense of weak convergence, we have $u(t_{n_k}) \to x$ weakly in R_2. On the other hand, since $u(t) \in \mathbb{C}([0, T_0]; R_2)$, it follows that $u(t_{n_k}) \to u(T_0)$ strongly (and, therefore, weakly) in R_2. Thus $x = u(T_0)$. Since x belongs to R_1 (being the weak limit of a sequence of elements of R_1), we have $u(T_0) \in R_1$, as required. Hence,

$$u(T_0) \in R_1. \tag{6.98}$$

Remark 6.7. A modification of our proof of (6.98) shows that the function $u(t)\colon [0, T] \to R_1$ is weakly continuous on $[0, T]$. (Although not necessary for our purposes here, this fact is of interest in itself.) Indeed, take an arbitrary point $t_0 \in [0, T]$ and suppose that $t_n \to t_0$, $t_n \in E$. Using the continuity of the function $u(t)\colon [0, T] \to R_2$ and arguing as above, we obtain a subsequence $\{t_{n_k}\}$ such that $u(t_{n_k}) \to u(t_0) \in R_1$ weakly in R_1. (In particular, it follows that $\|u(t_0)\|_{R_1} \leqslant \|u(t)\|_{\mathbb{L}^\infty(0,T;R_1)}$.) We claim that the same limit property holds for the whole sequence $\{t_n\}$. Indeed, otherwise there would be a subsequence $\{t_{n_l}\}$ and an element $f \in R_1^*$ such that $\langle f, u(t_{n_l})\rangle_{R_1} \not\to \langle f, u(t_0)\rangle_{R_1}$. Hence we would have a subsequence $\{t_{n_{l_p}}\}$ and a number $c > 0$ such that

$$|\langle f, u(t_{n_{l_p}})\rangle_{R_1} - \langle f, u(t_0)\rangle_{R_1}| > c.$$

On the other hand, it was proved above that every sequence $\{u(t_k)\}$ with $t_k \to t_0$ contains a subsequence weakly converging to $u(t_0)$. Applying this assertion to $\{u(t_{n_{l_p}})\}$, we arrive at a contradiction.

Take any $T' \in (0, T_0)$. By our result on local solubility in the weak generalized sense, there is a $T^* = T^*(T') > 0$ such that the solution of the problem (3.1), (3.2) exists for $t \in (T', T' + T^*)$. Note that condition (6.98) holds. Hence there is an exact lower bound

$$T^{**} = \inf_{T' \in [0, T_0]} T^* > 0.$$

Putting

$$T' = T_0 - \frac{T^{**}}{2},$$

we get the following continuation of the solution:

$$\hat{u} = \bigl\{u(t),\ t \in [0,T'];\ u(t-T'),\ t \in [T',T'+T^{**}]\bigr\}.$$

Thus we obtain that

$$\hat{u} = \bigl\{u(t),\ t \in [0,T'];\ u(t-T'),\ t \in [T',T_0+T^{**}/2]\bigr\}.$$

Hence we are able to continue the solution beyond the time $T_0 > 0$, a contradiction. Therefore the limit property (6.84) holds.

Thus we have proved the following theorem.

Theorem 6.8. *In addition to the conditions on the operator-valued coefficients stated above, we now adopt conditions* (6.7), (6.14), (6.22) (*where, moreover,* $\mathbb{B} \hookrightarrow\hookrightarrow \mathbb{V}^*$), *condition* (6.81) *and the conditions*

$$\mathbb{W}_1 \hookrightarrow\hookrightarrow \mathbb{W}_2 \subset \mathbb{V}, \quad \mathbb{W}_1 \hookrightarrow\hookrightarrow \mathbb{V}_j \subset \mathbb{V}, \quad \mathbb{V}_0 \subset \mathbb{V}, \quad \mathbb{V}_0 \subset \mathbb{W}_2, \qquad j=0,\dots,n,$$
$$(\mathbb{H}'_f(v_1) - \mathbb{H}'_f(v_2), v_1 - v_2)_1 \geqslant a|v_1 - v_2|_1^p, \qquad v_1, v_2 \in \mathbb{W}_1.$$

Suppose that the Hilbert space $\overline{\overline{\mathbb{W}}} \overset{\mathrm{ds}}{\subset} \mathbb{W}$ *has an orthonormal basis of eigenvectors* w_j *of the operator* $\mathbb{A}$. *Then the following assertions hold. If* $p \geqslant 2q+2$, *then a weak generalized solution exists for all* $T \in (0,+\infty)$. *If* $p < 2q+2$, *then there is a* $T_0 > 0$ *such that a weak generalized solution exists for all* $T \in (0,T_0)$. *Here either* $T_0 = +\infty$, *or* $T_0 < +\infty$ *and, in the latter case, we have*

$$\limsup_{t \uparrow T_0} \bigl(|u|_1(t) + \|u'\|(t)\bigr) = +\infty. \tag{6.99}$$

§ 7. Blow-up of weak generalized solutions

Sufficient conditions for the finite-time blow-up of weak generalized solutions cannot be obtained from the original statement of the problem since such solutions are not sufficiently smooth. However, the Galerkin approximants constructed in § 6 possess the required smoothness: $u_m(t) \in \mathbb{C}^{(2)}([0,T];\mathbb{W}_1)$.

We introduce the following notation:

$$\Phi_m(t) = \frac{1}{2}\langle \mathbb{A}u_m, u_m\rangle + \int_0^t \biggl[\frac{1}{2}\langle \mathbb{A}_0 u_m, u_m\rangle_0 + \sum_{j=1}^n \frac{p_j-1}{p_j}\langle \mathbb{A}_j(u_m), u_m\rangle_j\biggr]\,ds$$
$$+ \frac{1}{2p_0}\langle \mathbb{A}_0 u_{m0}, u_{m0}\rangle_0 + \frac{1}{p_0}\sum_{j=1}^n \frac{p_j-1}{p_j}\langle \mathbb{A}_j(u_{m0}), u_{m0}\rangle_j, \tag{7.1}$$

$$J_m(t) = \langle \mathbb{A}u'_m, u'_m\rangle + \int_0^t \biggl[\langle \mathbb{A}_0 u'_m, u'_m\rangle_0 + \sum_{j=1}^n \langle (\mathbb{A}_j(u_m))', u'_m\rangle_j\biggr]\,ds$$
$$+ \frac{1}{2}\langle \mathbb{A}_0 u_{m0}, u_{m0}\rangle_0 + \sum_{j=1}^n \frac{p_j-1}{p_j}\langle \mathbb{A}_j(u_{m0}), u_{m0}\rangle_j, \tag{7.2}$$

where $p_0 = \max_{j=1,\dots,n} p_j$. As in the proof of Lemma 5.1, we obtain that

$$(\Phi'_m)^2 \leqslant p_0 \Phi_m J_m, \qquad t \in [0,T]. \tag{7.3}$$

Regarding the Galerkin approximation problem (6.1)–(6.3) as the original problem for u_m and arguing as in § 5, we obtain the differential inequality

$$\Phi_m(t) \geqslant \frac{1}{[\Phi_m^{1-\alpha}(0) - A_m t]^{1/(\alpha-1)}}, \tag{7.4}$$

where

$$A_m^2 = (\alpha-1)^2 \Phi_m^{-2\alpha}(0)\big[(\Phi_m'(0))^2 - \delta_m \Phi_m(0)\big] > 0,$$

$$\delta_m = \begin{cases} \dfrac{2E_m(0)}{2\alpha-1} & \text{for } \ E(0) > 0, \\ 0 & \text{for } \ E(0) \leqslant 0, \end{cases} \qquad \alpha = \frac{1}{p_0}\left(1 + \frac{\theta}{2}\right), \quad p_0 = \max_{j=1,\dots,n} p_j,$$

$$\Phi_m(0) = \frac{1}{2}\langle \mathbb{A} u_{m0}, u_{m0}\rangle + \frac{1}{2p_0}\langle \mathbb{A}_0 u_{m0}, u_{m0}\rangle_0 + \frac{1}{p_0}\sum_{j=1}^n \frac{p_j-1}{p_j}\langle \mathbb{A}_j(u_{m0}), u_{m0}\rangle_j,$$

$$\Phi_m'(0) = \langle \mathbb{A} u_{m0}, u_{m1}\rangle + \frac{1}{2}\langle \mathbb{A}_0 u_{m0}, u_{m0}\rangle_0 + \sum_{j=1}^n \frac{p_j-1}{p_j}\langle \mathbb{A}_j(u_{m0}), u_{m0}\rangle_j,$$

$$\begin{aligned} E_m(0) &= \frac{\theta}{2}\langle \mathbb{A} u_{m1}, u_{m1}\rangle + \theta \mathbb{H}(u_{m0}) \\ &\quad + \left(1 + \frac{\theta}{2}\right)\left(\frac{1}{2}\langle \mathbb{A}_0 u_{m0}, u_{m0}\rangle_0 + \sum_{j=1}^n \frac{p_j-1}{p_j}\langle \mathbb{A}_j(u_{m0}), u_{m0}\rangle_j\right) - \theta \mathbb{F}(u_{m0}). \end{aligned}$$

We now let $m \to +\infty$ in (7.4). First, by Lemma 6.6 we have

$$\lim_{m\to+\infty} \langle \mathbb{A} u_m, u_m\rangle = \langle \mathbb{A} u, u\rangle \quad \text{uniformly with respect to } \ t \in [0,T]. \tag{7.5}$$

Moreover, by what was proved above,

$$u_m \to u \quad \text{strongly in } \ \mathbb{L}^{p_j}(0,T;\mathbb{V}_j), \tag{7.6}$$

$$\mathbb{A}_j(u_m) \to \mathbb{A}_j(u) \quad \text{strongly in } \ \mathbb{L}^{p_j'}(0,T;\mathbb{V}_j^*) \tag{7.7}$$

for $j = 0, \dots, n$. It follows that

$$\langle \mathbb{A}_j(u_m), u_m\rangle_j \to \langle \mathbb{A}_j(u), u\rangle_j \quad \text{for almost all } \ t \in [0,T], \tag{7.8}$$

and the sequence of numbers $\{\langle \mathbb{A}_j(u_m), u_m\rangle_j\}$ is bounded uniformly with respect to $m \in \mathbb{N}$. Using (7.5)–(7.8) and the initial conditions (6.2), (6.3), we conclude that the following limit property holds pointwise for $t \in [0,T]$:

$$\Phi_m(t) \to \Phi(t), \qquad t \in [0,T], \tag{7.9}$$

where

$$\begin{aligned} \Phi(t) = \frac{1}{2}\langle \mathbb{A} u, u\rangle + \int_0^t \left(\frac{1}{2}\langle \mathbb{A}_0 u, u\rangle_0 + \sum_{j=1}^n \frac{p_j-1}{p_j}\langle \mathbb{A}_j(u), u\rangle_j\right) ds \\ + \frac{1}{2p_0}\langle \mathbb{A}_0 u_0, u_0\rangle_0 + \frac{1}{p_0}\sum_{j=1}^n \frac{p_j-1}{p_j}\langle \mathbb{A}_j(u_0), u_0\rangle_j. \end{aligned}$$

The initial conditions (6.2) and (6.3) enable us to obtain that

$$
\begin{gathered}
A_m \to A, \qquad (7.10)\\
A^2 = (\alpha-1)^2\Phi^{-2\alpha}(0)\big[(\Phi'(0))^2 - \delta\Phi(0)\big] > 0,\\
\delta = \begin{cases} \dfrac{2E(0)}{2\alpha-1} & \text{for} \quad E(0) > 0,\\ 0 & \text{for} \quad E(0) \leqslant 0, \end{cases} \qquad \alpha = \frac{1}{p_0}\left(1+\frac{\theta}{2}\right), \quad p_0 = \max_{j=1,\dots,n} p_j,\\
\Phi(0) = \frac{1}{2}\langle \mathbb{A}u_0, u_0\rangle + \frac{1}{2p_0}\langle \mathbb{A}_0 u_0, u_0\rangle_0 + \frac{1}{p_0}\sum_{j=1}^{n}\frac{p_j-1}{p_j}\langle \mathbb{A}_j(u_0), u_0\rangle_j,\\
\Phi'(0) = \langle \mathbb{A}u_0, u_1\rangle + \frac{1}{2}\langle \mathbb{A}_0 u_0, u_0\rangle_0 + \sum_{j=1}^{n}\frac{p_j-1}{p_j}\langle \mathbb{A}_j(u_0), u_0\rangle_j,
\end{gathered}
$$

$$
\begin{aligned}
E(0) = {} & \frac{\theta}{2}\langle \mathbb{A}u_1, u_1\rangle + \theta\mathbb{H}(u_0)\\
& + \left(1+\frac{\theta}{2}\right)\left(\frac{1}{2}\langle \mathbb{A}_0 u_0, u_0\rangle_0 + \sum_{j=1}^{n}\frac{p_j-1}{p_j}\langle \mathbb{A}_j(u_0), u_0\rangle_j\right) - \theta\mathbb{F}(u_0).
\end{aligned}
$$

Therefore in the limit we obtain from (7.4) that

$$
\Phi(t) \geqslant \frac{1}{[\Phi^{1-\alpha}(0) - At]^{1/(\alpha-1)}}. \qquad (7.11)
$$

Thus we have proved the following theorem.

Theorem 7.1. *Suppose that $p < 2q+2$ and the hypotheses of Theorem 6.8 hold. If the following conditions hold*:

$$
\Phi'(0) > (\delta\Phi(0))^{1/2}, \quad \Phi(0) > 0, \quad \theta > 2(p_0-1), \quad \theta \geqslant \mu, \qquad (7.12)
$$

$$
\delta = \begin{cases} \dfrac{2E(0)}{2\alpha-1} & \textit{for} \quad E(0) > 0,\\ 0 & \textit{for} \quad E(0) \leqslant 0, \end{cases} \qquad \alpha = \frac{1}{p_0}\left(1+\frac{\theta}{2}\right), \quad p_0 = \max_{j=1,\dots,n} p_j,
$$

then the time $T > 0$ cannot be arbitrarily large. Namely, we have

$$
\begin{gathered}
T \leqslant T_\infty \leqslant \Phi^{1-\alpha}(0)A^{-1},\\
A^2 \equiv (\alpha-1)^2\Phi^{-2\alpha}(0)[(\Phi'(0))^2 - \delta\Phi(0)],
\end{gathered}
$$

and (7.11) holds for all $t \in [0, T_\infty)$.

Remark 7.2. We claim that $T_0 < +\infty$ under the hypotheses of Theorem 7.1. Indeed, assume that $T_0 = +\infty$. Then

$$
|u|_1(t) + \|u'\| \leqslant c(T) < +\infty, \qquad t \in [0, T],
$$

for all $T \in (0, +\infty)$. Since there are embeddings

$$
\mathbb{W}_1 \subset \mathbb{V}, \quad \mathbb{W}_1 \subset \mathbb{V}_j, \qquad j = 0, \dots, n,
$$

we conclude that

$$\Phi(t) \leqslant c(T) < +\infty, \qquad t \in [0,T],$$

for all $T \in (0,+\infty)$. This contradiction shows that $T_0 < +\infty$ under the hypotheses of Theorem 7.1. Hence the limit equality (6.99) also holds.

§ 8. Local solubility in the sense of strong generalized solutions, and their blow-up

First of all we make a basic assumption: suppose that $\mathbb{W} = \mathbb{V}$. We understand solutions of the problem (3.1), (3.2) in the weak generalized sense of Definition 3.3. For convenience we reproduce the problem here:

$$\mathbb{A}\frac{d^2u}{dt^2} + \frac{d}{dt}\biggl(\mathbb{A}_0 u + \sum_{j=1}^{n} \mathbb{A}_j(u)\biggr) + \mathbb{H}'_f(u) - \mathbb{F}'_f(u) = \theta \in \mathbb{V}^*, \tag{8.1}$$

$$u(0) = u_0 \in \mathbb{V}, \qquad u'(0) = u_1 \in \mathbb{V}. \tag{8.2}$$

By conditions $\mathbb{A}$ and the Browder–Minty theorem, there is an inverse operator

$$\mathbb{A}^{-1}\colon \mathbb{V}^* \to \mathbb{V}, \quad \|\mathbb{A}^{-1}v_1 - \mathbb{A}^{-1}v_2\|^* \leqslant \frac{1}{\mathrm{m}}\|v_1 - v_2\|, \qquad v_1, v_2 \in \mathbb{V}.$$

We put $v = \mathbb{A}u$. Then the problem (8.1), (8.2) for a function $u(t)$ in the class $\mathbb{C}^{(2)}([0,T];\mathbb{V})$ (the operators d^2/dt^2 and $\mathbb{A}$ commute on this class of functions) becomes a Cauchy problem of the form

$$\frac{d^2v}{dt^2} + \frac{d}{dt}\biggl(\mathbb{A}_0\mathbb{A}^{-1}v + \sum_{j=1}^{n} \mathbb{A}_j(\mathbb{A}^{-1}v)\biggr) + \mathbb{H}'_f(\mathbb{A}^{-1}v) - \mathbb{F}'_f(\mathbb{A}^{-1}v) = \theta \in \mathbb{V}^*, \tag{8.3}$$

$$v_0 = v(0) = \mathbb{A}u_0, \qquad v_1 = v'(0) = \mathbb{A}u_1. \tag{8.4}$$

Note that in the class $\mathbb{C}^{(2)}([0,T];\mathbb{V}^*)$ equation (8.3) can be written as

$$\begin{aligned} v = H(v) \equiv v_0 + v_1 t + \biggl(\mathbb{A}_0\mathbb{A}^{-1}v_0 + \sum_{j=1}^{n} \mathbb{A}_j(\mathbb{A}^{-1}v_0)\biggr)t \\ + \int_0^t (t-s)[\mathbb{F}'_f(\mathbb{A}^{-1}v) - \mathbb{H}'_f(\mathbb{A}^{-1}v)]\,ds - \int_0^t \biggl[\mathbb{A}_0\mathbb{A}^{-1}v + \sum_{j=1}^{n} \mathbb{A}_j(\mathbb{A}^{-1}v)\biggr]\,ds. \end{aligned} \tag{8.5}$$

We now introduce the following closed, bounded and convex subset of the Banach space $\mathbb{L}^\infty(0,T;\mathbb{V}^*)$:

$$\mathbb{V}_r \equiv \Bigl\{v \in \mathbb{L}^\infty(0,T;\mathbb{V}^*)\colon |||v||| \equiv \operatorname*{ess.sup}_{t\in[0,T]} \|v\|^* \leqslant r\Bigr\}.$$

We claim that if $r > 0$ is sufficiently large, $T > 0$ is sufficiently small, and

$$\|v_0\|^* \leqslant \frac{r}{2}, \tag{8.6}$$

then the operator $H(\,\cdot\,)$ maps $\mathbb{V}_r$ to itself. Indeed, the following estimate holds:

$$|||H(v)||| \leqslant \|v_0\|^* + T\biggl[\frac{M_0}{\mathrm{m}}\|v_0\|^* + \sum_{j=1}^{n}\frac{M_j}{\mathrm{m}^{p_j-1}}(\|v_0\|^*)^{p_j-1} + \|v_1\|^*\biggr] \\ + T^2\biggl[\frac{M_F}{\mathrm{m}^{q+1}}|||v|||^{q+1} + \frac{M_H}{\mathrm{m}^{p-1}}|||v|||^{p-1}\biggr] + T\biggl[\frac{M}{\mathrm{m}}|||v||| + \sum_{j=1}^{n}\frac{M_j}{\mathrm{m}^{p_j-1}}|||v|||^{p_j-1}\biggr]. \tag{8.7}$$

We fix $r > 0$ so large that (8.6) holds. Let $T > 0$ be so small that

$$|||H(v)||| \leqslant r, \qquad v \in \mathbb{V}_r.$$

Then $H(\,\cdot\,)\colon \mathbb{V}_r \to \mathbb{V}_r$, as required. We now claim that this operator is a contraction mapping on $\mathbb{V}_r$. Indeed, the following estimate holds:

$$|||H(\bar{v}) - H(\bar{\bar{v}})||| \leqslant \biggl[\frac{a_1T^2}{\mathrm{m}}\mu_1(R_1) + \frac{a_2T^2}{\mathrm{m}}\mu_2(R_2) + \frac{a_3TM}{\mathrm{m}} \\ + T\sum_{j=1}^{n}\frac{\overline{M}_j a_{4j}}{\mathrm{m}^{p_j-1}}(R_j^*)^{p_j-2}\biggr]|||\bar{v} - \bar{\bar{v}}|||, \tag{8.8}$$

where

$$R_1 = \frac{a_1}{\mathrm{m}}\max\{|||\bar{v}|||, |||\bar{\bar{v}}|||\}, \qquad R_2 = \frac{a_2}{\mathrm{m}}\max\{|||\bar{v}|||, |||\bar{\bar{v}}|||\},$$
$$R_j^* = a_{4j}\max\{|||\bar{v}|||, |||\bar{\bar{v}}|||\}$$

and a_1 is the constant in the continuous embedding $\mathbb{W}_2^* \subset \mathbb{V}^*$, a_2 is the constant in the continuous embedding $\mathbb{W}_1^* \subset \mathbb{V}^*$, a_3 is the constant in the continuous embedding $\mathbb{V}_0^* \subset \mathbb{V}^*$ and a_{4j} is the constant in the continuous embedding $\mathbb{V}_j^* \subset \mathbb{V}^*$. Thus, for sufficiently large $r > 0$ and sufficiently small $T > 0$, (8.8) yields the desired inequality

$$|||H(\bar{v}) - H(\bar{\bar{v}})||| \leqslant \frac{1}{2}|||\bar{v} - \bar{\bar{v}}||| \quad \forall\, \bar{v}, \bar{\bar{v}} \in \mathbb{V}_r.$$

By the contraction mapping theorem, the integral equation (8.5) has a unique solution $v(t) \in \mathbb{V}_r$. Continuing this solution in time by the standard method, we obtain a maximal $T_0 > 0$ such that for all $T \in (0, T_0)$ there is a solution of class $\mathbb{L}^\infty(0, T; \mathbb{V}^*)$. Here either $T_0 = +\infty$, or $T_0 < +\infty$ and, in the latter case, we have

$$\lim_{T\uparrow T_0}\sup_{t\in[0,T]}\|v\|^*(t) = +\infty.$$

Using the 'bootstrap' method in the standard way, we obtain from (8.5) that $v(t) \in \mathbb{C}^{(2)}([0, T_0); \mathbb{V}^*)$.

Finally, we conclude from the Lipschitz continuity of $\mathbb{A}^{-1}$ that $u(t) \in \mathbb{C}^{(2)}([0, T_0); \mathbb{V})$. This proves the following theorem.

Theorem 8.1. *For all $u_0, u_1 \in \mathbb{V}$ there is a unique strong generalized solution $u(t)$ of the problem* (8.1), (8.2) *of class $\mathbb{C}^{(2)}([0, T_0); \mathbb{V})$ for some $T_0 > 0$. Here either $T_0 = +\infty$ or $T_0 < +\infty$ and, in the latter case, we have*

$$\lim_{T\uparrow T_0}\sup_{t\in[0,T]}\|\mathbb{A}u\|(t) = +\infty. \tag{8.9}$$

Sufficient conditions for the blow-up of the strong generalized solution of the problem (8.1), (8.2) were obtained in Theorem 2.1.

Theorem 8.2. *Suppose that the hypotheses of Theorem 2.1 hold. Then the time $T_0 > 0$ in Theorem 8.1 is bounded and, therefore, the limit property (8.9) holds.*

Proof. Suppose that the hypotheses of Theorem 2.1 hold. Then that theorem yields that $T_0 < +\infty$. Hence it follows from Theorem 8.1 that the second case of the alternative holds. $\square$

§ 9. Examples

First of all we note that the procedure of obtaining sufficient conditions for the blow-up of solutions does not depend on the boundedness of the domain and, therefore, is applicable to the corresponding Cauchy problems. But local solubility is often available only for bounded domains. In all the examples below, the proof of blow-up works for the Cauchy problems. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with sufficiently smooth boundary. As usual, we write

$$p^* = \begin{cases} \dfrac{Np}{N-p} & \text{for} \quad N > p, \\ +\infty & \text{for} \quad N \leqslant p. \end{cases}$$

In the examples below we also put

$$\mathbb{L} = \mathbb{L}^2(\Omega), \qquad a_j(x) \in \mathbb{L}^\infty(\Omega), \qquad 0 < a_0 \leqslant a_j(x).$$

Example 9.1. Consider the problem

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial t}\biggl(-\Delta u + \sum_{j=1}^n a_j(x)|u|^{p_j-2}u\biggr) - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = |u|^q u, \tag{9.1}$$

$$u(x,0) = u_0(x) \in \mathbb{W}_0^{1,p}(\Omega), \qquad u'(x,0) = u_1(x) \in \mathbb{L}^2(\Omega), \tag{9.2}$$

$$u|_{\partial\Omega} = 0. \tag{9.3}$$

In this case we have

$$\mathbb{A} = I, \qquad \mathbb{A}_0 = -\Delta, \qquad \mathbb{A}_j(u) = a_j(x)|u|^{p_j-2}u,$$
$$\mathbb{H}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx, \qquad \mathbb{F}(u) = \frac{1}{q+2}\int_\Omega |u|^{q+2}\,dx.$$

Here and in what follows, I is the identity operator. Moreover, suppose that

$$\mathbb{V} = \mathbb{L}^2(\Omega), \qquad \mathbb{V}_0 = \mathbb{H}_0^1(\Omega), \qquad \mathbb{V}_j = \mathbb{L}^{p_j}(\Omega),$$
$$\mathbb{W}_1 = \mathbb{W}_0^{1,p}(\Omega), \qquad \mathbb{W}_2 = \mathbb{L}^{q+2}(\Omega), \qquad \mathbb{B} = \mathbb{H}_0^1(\Omega),$$

$\overline{\mathbb{W}} = \mathbb{H}_0^s(\Omega)$ for sufficiently large $s > 1$ and $\{w_k\} \subset \overline{\mathbb{W}}$ are solutions of the problem

$$(-\Delta)^s w_k = \lambda_k w_k, \qquad \mathbb{J} = (-\Delta)^{-s} \colon \mathbb{H}^{-s}(\Omega) \to \mathbb{H}_0^s(\Omega).$$

Therefore condition (6.14) holds. Thus, assuming that

$$q+2<p^*, \quad p_j<p^*, \qquad j=1,\dots,n,$$

we have completely continuous embeddings

$$\mathbb{W}_1 \hookrightarrow\hookrightarrow \mathbb{W}_2, \quad \mathbb{W}_1 \hookrightarrow\hookrightarrow \mathbb{V}_j, \qquad j=1,\dots,n.$$

Clearly,

$$\mathbb{W}_2=\mathbb{L}^{q+2}(\Omega)\subset\mathbb{L}^2(\Omega)=\mathbb{V}$$

for $q>0$ and $\mathbb{V}_j=\mathbb{L}^{p_j}(\Omega)\subset\mathbb{L}^2(\Omega)=\mathbb{V}$. Moreover,

$$\mathbb{V}_0=\mathbb{H}_0^1(\Omega)\subset\mathbb{L}^2(\Omega)=\mathbb{V}.$$

Finally, if $q+2\leqslant 2^*$, then we have the embedding

$$\mathbb{V}_0=\mathbb{H}_0^1(\Omega)\subset\mathbb{L}^{q+2}(\Omega)=\mathbb{W}_2.$$

In the case when $2q+2\leqslant 2^*$, property (6.7) holds. Properties (6.22) and (6.81) also hold. Therefore Theorem 6.8 is applicable, and the blow-up of solutions occurs under the hypotheses of Theorem 7.1.

Example 9.2. Consider the following problem:

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial}{\partial t}\left(\Delta^2 u-\sum_{j=1}^n \operatorname{div}(a_j(x)|\nabla u|^{p_j-2}\nabla u)\right) \\ +\Delta(|\Delta u|^{p-2}\Delta u)=-\operatorname{div}(|\nabla u|^q\nabla u), \tag{9.4}$$

$$u(x,0)=u_0(x)\in\mathbb{W}_0^{2,p}(\Omega), \qquad u'(x,0)=u_1(x)\in\mathbb{L}^2(\Omega), \tag{9.5}$$

$$u|_{\partial\Omega}=0, \qquad \left.\frac{\partial u}{\partial n_x}\right|_{\partial\Omega}=0. \tag{9.6}$$

In this case we have

$$\mathbb{A}=I, \qquad \mathbb{A}_0=\Delta^2, \qquad \mathbb{A}_j(u)=-\operatorname{div}(a_j(x)|\nabla u|^{p_j-2}\nabla u),$$
$$\mathbb{H}(u)=\frac{1}{p}\int_\Omega |\Delta u|^p\,dx, \qquad \mathbb{F}(u)=\frac{1}{q+2}\int_\Omega|\nabla u|^{q+2}\,dx.$$

Here

$$\mathbb{V}=\mathbb{L}^2(\Omega), \qquad \mathbb{V}_0=\mathbb{H}_0^2(\Omega), \qquad \mathbb{V}_j=\mathbb{W}_0^{1,p_j}(\Omega),$$
$$\mathbb{W}_1=\mathbb{W}_0^{2,p}(\Omega), \qquad \mathbb{W}_2=\mathbb{W}_0^{1,q+2}(\Omega), \qquad \mathbb{B}=\mathbb{H}_0^2(\Omega).$$

Moreover, $\overline{\mathbb{W}}=\mathbb{H}_0^s(\Omega)$ for sufficiently large $s>0$, and $\{w_k\}\subset\overline{\mathbb{W}}$ are solutions of the problem

$$(-\Delta)^s w_k=\lambda_k w_k, \qquad \mathbb{J}=(-\Delta)^{-s}\colon \mathbb{H}^{-s}(\Omega)\to\mathbb{H}_0^s(\Omega).$$

Therefore property (6.14) holds.

Thus, assuming that

$$q + 2 < p^*, \qquad p_j < p^*$$

for $j = 0, \ldots, n$, we have completely continuous embeddings

$$\mathbb{W}_1 \hookrightarrow\hookrightarrow \mathbb{W}_2, \quad \mathbb{W}_1 \hookrightarrow\hookrightarrow \mathbb{V}_j, \qquad j = 0, \ldots, n.$$

Clearly,

$$\mathbb{V}_0 = \mathbb{H}_0^2(\Omega) \subset \mathbb{L}^2(\Omega) = \mathbb{V}.$$

If

$$q + 2 \leqslant 2^*,$$

then there is a continuous embedding

$$\mathbb{V}_0 = \mathbb{H}_0^2(\Omega) \subset \mathbb{W}_0^{1,q+2}(\Omega) = \mathbb{W}_2.$$

Finally, if

$$2q + 2 \leqslant 2^*,$$

then property (6.7) holds. Properties (6.22) and (6.81) also hold. Therefore Theorem 6.8 is applicable, and the blow-up of solutions occurs under the hypotheses of Theorem 7.1.

Example 9.3. Consider the following problem:

$$\begin{aligned} -\Delta\frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial t}\biggl(\Delta^2 u - \sum_{j=1}^{n} \operatorname{div}(a_j(x)|\nabla u|^{p_j-2}\nabla u)\biggr) + \Delta^2 u \\ = -\operatorname{div}(|\nabla u|^q \nabla u), \end{aligned} \tag{9.7}$$

$$u(x,0) = u_0(x) \in \mathbb{H}^2(\Omega) \cap \mathbb{H}_0^1(\Omega), \qquad u'(x,0) = u_1(x) \in \mathbb{H}_0^1(\Omega), \tag{9.8}$$

$$u|_{\partial\Omega} = 0, \qquad \Delta u|_{\partial\Omega} = 0. \tag{9.9}$$

In this case we have

$$\mathbb{A} = -\Delta, \qquad \mathbb{A}_0 = \Delta^2, \qquad \mathbb{A}_j(u) = -\operatorname{div}(a_j(x)|\nabla u|^{p_j-2}\nabla u),$$
$$\mathbb{H}(u) = \frac{1}{2}\int_\Omega |\Delta u|^2\,dx, \qquad \mathbb{F}(u) = \frac{1}{q+2}\int_\Omega |\nabla u|^{q+2}\,dx.$$

Here

$$\mathbb{V} = \mathbb{H}_0^1(\Omega), \qquad \mathbb{V}_0 = \{u \in \mathbb{H}^2(\Omega) \cap \mathbb{H}_0^1(\Omega) \colon u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0\},$$
$$\mathbb{V}_j = \mathbb{W}_0^{1,p_j}(\Omega), \qquad \mathbb{W}_1 = \{u \in \mathbb{H}^2(\Omega) \cap \mathbb{H}_0^1(\Omega) \colon u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0\},$$
$$\mathbb{W}_2 = \mathbb{W}_0^{1,q+2}(\Omega), \qquad \mathbb{B} = \mathbb{L}^2(\Omega).$$

Thus, if $q + 2 < 2^*$ and $p_j < 2^*$ for $j = 1, \ldots, n$, then we have completely continuous embeddings

$$\mathbb{W}_1 \hookrightarrow\hookrightarrow \mathbb{W}_2, \quad \mathbb{W}_1 \hookrightarrow\hookrightarrow \mathbb{V}_j, \qquad j = 1, \ldots, n.$$

Moreover,

$$\mathbb{W}_2 = \mathbb{W}_0^{1,q+2}(\Omega) \subset \mathbb{H}_0^1(\Omega) = \mathbb{V}, \qquad \mathbb{V}_j = \mathbb{W}_0^{1,p_j}(\Omega) \subset \mathbb{H}_0^1(\Omega) = \mathbb{V},$$
$$\mathbb{V}_0 \subset \mathbb{H}_0^1(\Omega) = \mathbb{V}, \qquad \mathbb{V}_0 \subset \mathbb{W}_2, \qquad q+2 \leqslant 2^*.$$

Finally, if $2q+2 \leqslant 2^*$, then property (6.7) holds. Property (6.22) also holds. The following remark shows that the operator $\mathbb{A}$ possesses all the properties that were used in our proof of local solubility in the weak generalized sense.

Remark 9.4. It is known that the space $\mathbb{H}_0^1(\Omega) \cap \mathbb{H}^2(\Omega)$ admits a basis (in the sense of convergence in the $\mathbb{H}_0^2(\Omega)$-norm) of eigenvectors of the Laplace operator provided that the boundary of Ω is sufficiently smooth (see, for example, [17], Russian pp. 132–133). It is also known (see [17], Russian p. 119, or [18], Russian p. 44) that the expression $\|\Delta u\|_{\mathbb{L}^2(\Omega)}$ is a norm on $\mathbb{H}_0^1(\Omega) \cap \mathbb{H}^2(\Omega)$ equivalent to the standard $\mathbb{H}_0^2$-norm. By choosing $(\Delta u, \Delta v)_{\mathbb{L}^2(\Omega)}$ as the scalar product in $\mathbb{H}_0^1(\Omega) \cap \mathbb{H}^2(\Omega)$ and appropriately normalizing the system $\{v_n\}$ (orthogonal in $\mathbb{L}^2(\Omega)$) of eigenfunctions of the Laplace operator in Ω, we get the required orthogonal basis because

$$(\Delta v_n, \Delta v_m)_{\mathbb{L}^2(\Omega)} = \lambda_n \lambda_m (v_n, v_m)_{\mathbb{L}^2(\Omega)} = 0.$$

We now prove property (6.81). First, there is a chain of continuous embeddings, one of which is dense:

$$\mathbb{H}_0^1(\Omega) \cap \mathbb{H}^2(\Omega) \overset{\mathrm{ds}}{\subset} \mathbb{H}_0^1(\Omega) \subset \mathbb{H}^{-1}(\Omega) \subset (\mathbb{H}_0^1(\Omega) \cap \mathbb{H}^2(\Omega))^*. \tag{9.10}$$

In the case under consideration,

$$\overline{\mathbb{W}} = \mathbb{H}_0^1(\Omega) \cap \mathbb{H}^2(\Omega).$$

Note that

$$\Delta\colon \mathbb{H}_0^1(\Omega) \to \mathbb{H}^{-1}(\Omega). \tag{9.11}$$

We denote the duality brackets between $\overline{\mathbb{W}}$ and $\overline{\mathbb{W}}^*$ by $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle_{\overline{\mathbb{W}}}$. Using the dense embedding in (9.10) along with (9.11), we conclude that there is an equation of duality brackets:

$$\langle -\Delta u, u\rangle = \langle\langle -\Delta u, u\rangle\rangle_{\overline{\mathbb{W}}}, \tag{9.12}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ are the duality brackets between $\mathbb{V} = \mathbb{H}_0^1(\Omega)$ and $\mathbb{V}^* = \mathbb{H}^{-1}(\Omega)$. Note that the following chain of equalities holds for every $z \in \mathbb{V} \subset \overline{\mathbb{W}}^*$:

$$\begin{aligned}\|\Delta z\|_{\overline{\mathbb{W}}}^* &= \sup_{\|\Delta w\|_{\mathbb{L}^2(\Omega)}=1} |\langle\langle \Delta z, w\rangle\rangle_{\overline{\mathbb{W}}}| = \sup_{\|\Delta w\|_{\mathbb{L}^2(\Omega)}=1} |\langle \Delta z, w\rangle| \\ &= \sup_{\|\Delta w\|_{\mathbb{L}^2(\Omega)}=1} \left|\int_\Omega (\nabla w, \nabla z)\,dx\right| = \sup_{\|\Delta w\|_{\mathbb{L}^2(\Omega)}=1} \left|\int_\Omega z\Delta w\,dx\right| = \|z\|_{\mathbb{L}^2(\Omega)}\end{aligned} \tag{9.13}$$

for all

$$w \in \overline{\mathbb{W}} = \mathbb{H}^2(\Omega) \cap \mathbb{H}_0^1(\Omega) \overset{\mathrm{ds}}{\subset} \mathbb{L}^2(\Omega).$$

The last equality in (9.13) is justified in the following way. It suffices to take w equal to a solution (up to normalization) of the homogeneous Dirichlet problem for the Poisson equation with right-hand side z. Since the boundary of Ω is sufficiently smooth (see Remark 9.4), this solution belongs to $\mathbb{H}^2(\Omega)\cap\mathbb{H}^1_0(\Omega)$. Clearly, there is a continuous embedding

$$\mathbb{L}^2(\Omega)\subset(\mathbb{H}^2(\Omega)\cap\mathbb{H}^1_0(\Omega))^*.$$

This proves that property (6.81) holds.

We now verify that condition (6.14) holds. Note that

$$\overline{\mathbb{W}}\equiv\bigl\{u\in\mathbb{H}^2(\Omega)\cap\mathbb{H}^1_0(\Omega)\colon u|_{\partial\Omega}=\Delta u|_{\partial\Omega}=0\bigr\}$$

and the dualizing map $\mathbb{J}$ between $\overline{\mathbb{W}}^*$ and $\overline{\mathbb{W}}$ is an integral operator whose kernel is the Green function of the following problem:

$$\Delta^2G(\,\cdot\,,y)=\delta(\,\cdot-y)\quad\text{in}\ \ \Omega,\qquad G(\,\cdot\,,y)=-\Delta G(\,\cdot\,,y)=0\quad\text{on}\ \ \partial\Omega.$$

It is known that the solution of this problem exists and can be written out explicitly. By our choice of the system of functions $\{w_k\}$, they are solutions of the problem

$$-\Delta w_k=\lambda_k w_k\quad\Longrightarrow\quad\Delta^2w_k=\lambda_k^2w_k,\qquad w_k|_{\partial\Omega}=\Delta w_k|_{\partial\Omega}=0.$$

Then the dualizing map $\mathbb{J}$ possesses the property

$$\mathbb{J}w_k=\frac{w_k}{\lambda_k^2}.$$

It follows that condition (6.14) holds.

Example 9.5. Consider the problem

$$(I-\Delta)\frac{\partial^2u}{\partial t^2}+\frac{\partial}{\partial t}\biggl(u+\sum_{j=1}^n a_j(x)|u|^{p_j-2}u\biggr)=|u|^qu,\qquad a_j(x)\in\mathbb{L}^\infty_+(\mathbb{R}^N),\tag{9.14}$$

$$u(0)=u_0\in\mathbb{H}^1(\mathbb{R}^N),\qquad u'(0)=u_1\in\mathbb{H}^1(\mathbb{R}^N).\tag{9.15}$$

In this case we have

$$\mathbb{A}=I-\Delta,\qquad\mathbb{A}_0=I,\qquad\mathbb{A}_j(u)=a_j(x)|u|^{p_j-2}u,$$
$$\mathbb{F}(u)=\frac{1}{q+2}\int_{\mathbb{R}^N}|u|^{q+2}\,dx,\qquad\mathbb{H}(u)=0,$$

where

$$\mathbb{V}=\mathbb{H}^1(\mathbb{R}^N),\quad\mathbb{V}_0=\mathbb{L}^2(\mathbb{R}^N),\quad\mathbb{V}_j=\mathbb{L}^{p_j}(\mathbb{R}^N),\quad\mathbb{W}_2=\mathbb{L}^{q+2}(\mathbb{R}^N).$$

We now require that the following conditions hold: $p_j\in(2,2^*]$, $q+2\in(2,2^*]$. This guarantees that the basic assumption in § 8 is satisfied:

$$\mathbb{W}=\mathbb{V}=\mathbb{H}^1(\mathbb{R}^N).$$

Thus all the hypotheses of Theorem 8.1 hold, and blow-up of solutions occurs under the hypotheses of Theorem 8.2.

Example 9.6. Consider the problem

$$(I+\Delta^2)\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial t}\biggl(\Delta u + \sum_{j=1}^{n} \operatorname{div}(a_j(x)|\nabla u|^{p_j-2}\nabla u)\biggr)$$
$$= -\operatorname{div}(|\nabla u|^q \nabla u), \qquad a_j(x) \in \mathbb{L}^\infty_+(\mathbb{R}^N), \tag{9.16}$$
$$u(0) = u_0 \in \mathbb{H}^2(\mathbb{R}^N), \qquad u'(0) = u_1 \in \mathbb{H}^2(\mathbb{R}^N). \tag{9.17}$$

In this case we have

$$\mathbb{A} = I + \Delta^2, \qquad \mathbb{A}_0 = -\Delta, \qquad \mathbb{A}_j(u) = -\operatorname{div}(a_j(x)|\nabla u|^{p_j-2}\nabla u),$$
$$\mathbb{F}(u) = \frac{1}{q+2}\int_{\mathbb{R}^N} |\nabla u|^{q+2}\,dx, \qquad \mathbb{H}(u) = 0,$$

where

$$\mathbb{V} = \mathbb{H}^2(\mathbb{R}^N), \quad \mathbb{V}_0 = \mathbb{H}^1(\mathbb{R}^N), \quad \mathbb{V}_j = \mathbb{W}^{1,p_j}(\mathbb{R}^N), \quad \mathbb{W}_2 = \mathbb{W}^{1,q+2}(\mathbb{R}^N).$$

We now require that the following conditions hold: $p_j \in (2, 2^*]$, $q+2 \in (2, 2^*]$. This guarantees that the basic assumption in § 8 is satisfied:

$$\mathbb{W} = \mathbb{V} = \mathbb{H}^2(\mathbb{R}^N).$$

Thus all the hypotheses of Theorem 8.1 hold, and blow-up of solutions occurs under the hypotheses of Theorem 8.2.

Example 9.7. Consider the following problem:

$$(I+\Delta^2)\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial t}\biggl(\Delta u + \sum_{j=1}^{n} \operatorname{div}(a_j(x)|\nabla u|^{p_j-2}\nabla u)\biggr)$$
$$= -\Delta u\biggl(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\biggr)^{q/2}, \qquad a_j(x) \in \mathbb{L}^\infty_+(\mathbb{R}^N), \tag{9.18}$$
$$u(0) = u_0 \in \mathbb{H}^2(\mathbb{R}^N), \qquad u'(0) = u_1 \in \mathbb{H}^2(\mathbb{R}^N). \tag{9.19}$$

In this case we have

$$\mathbb{A} = I + \Delta^2, \qquad \mathbb{A}_0 = -\Delta, \qquad \mathbb{A}_j(u) = -\operatorname{div}(a_j(x)|\nabla u|^{p_j-2}\nabla u),$$
$$\mathbb{F}(u) = \frac{1}{q+2}\biggl(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\biggr)^{(q+2)/2}, \qquad \mathbb{H}(u) = 0,$$

where

$$\mathbb{V} = \mathbb{H}^2(\mathbb{R}^N), \quad \mathbb{V}_0 = \mathbb{H}^1(\mathbb{R}^N), \quad \mathbb{V}_j = \mathbb{W}^{1,p_j}(\mathbb{R}^N), \quad \mathbb{W}_2 = \mathbb{H}^1(\mathbb{R}^N).$$

Suppose that the following conditions hold: $p_j \in (2, 2^*]$, $q > 0$. Then we obtain that the basic assumption in § 8 is satisfied:

$$\mathbb{W} = \mathbb{V} = \mathbb{H}^2(\mathbb{R}^N).$$

Thus all the hypotheses of Theorem 8.1 hold, and blow-up of solutions occurs under the hypotheses of Theorem 8.2.

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