If the number of revolutions in the interval T^* is designated as N, then from (45) and (46) we find

$$N=2nK/\pi\sigma.$$

In case "a" we will have, with the help of Eqs. (18) and (21),

$$N=BK/\pi\mu A(p'+q'), \quad (47)$$

where

$$\mu=\frac{m'}{m_0}, \quad B=\frac{16}{9}\left(\frac{q'}{a}\right)^3\left(\frac{1+e'}{1-e'}\right)^{3/2}$$

and m_0 is the mass of the central body.

The equations for the other cases can be written similarly. Thus, in case "d" we have

$$N=BK/2\pi\mu A\sqrt{pq}, \quad (48)$$

while in case "c" the equation for N has the same form as (47).

The behavior of N as a function of h for A = 0.1, e' = 0.6, and μ = 1 is shown in Fig. 4. It is seen from Fig. 4 that in the given case for the majority of initial conditions the period of variation of the eccentricity (left branch of the curve) and the hitting time (right branch of the curve) comprise 100-200 revolutions of body P about the central body P_0.

[1]E. P. Aksenov, Astron. Zh. 56 (1979) [Sov. Astron. 23, (1979)].
[2]Yu. S. Sikorskii, Elements of the Theory of Elliptic Functions with Applications to Mechanics [in Russian], Ob'ed. Nauch. Tekh. Izd. (1936).

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Poincaré periodic solutions of the third kind in the problem of the translational–rotational motion of a rigid body in the gravitational field of a sphere

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The existence of generating solutions which correspond to a new family of periodic solutions in the problem of the translational–rotational motion of a rigid body in the gravitational field of a sphere is demonstrated and a qualitative analysis of them is given. The solutions found are called solutions of the third kind and correspond to three-dimensional periodic motions of the bodies in a moving coordinate system rotating together with the line of nodes of the orbit in the Laplace plane. The analytical conditions for the existence of solutions of the third kind are studied for all possible cases of commensurability between the mean velocity of the orbital motion and the angular velocity of rotation of the body. A geometrical and dynamic interpretation of the corresponding generating solutions is given.

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INTRODUCTION

Reports of the author[2–4] have been devoted to the study of Poincaré periodic solutions[1] in the problem of the translational–rotational motion of two rigid bodies whose elementary particles interact by Newton's law. In these reports the investigation of periodic solutions is based on the use of equations of translational–rotational motion in Delaunay–Andoyer, canonical, osculating elements, which are obtained for the problem under consideration as a particular case from the general equations for the problem of n rigid bodies.[5]

The plane translational–rotational motion of a triaxial body in the gravitational field of a sphere was considered in Ref. 2. The periodic solutions found in that report actually comprise the periodic solutions of the second kind (in Poincaré's terminology) for the problem being considered here. In Ref. 3 the results were generalized to the case of the plane, translational–rotational motion of two rigid bodies possessing a common plane of dynamic symmetry. Periodic motions of a three-dimensional character in the problem of the motion of an axisymmetric rigid body in the gravitational field of a sphere were considered in Ref. 4.

In each of the enumerated reports the analytic conditions for the existence of Poincaré periodic solutions were obtained and they were analyzed qualitatively and numerically in the most interesting cases from a practical aspect.

In the present report such an analysis is extended to periodic solutions of the third kind in the problem of the translational–rotational motion of a rigid body in the gravitational field of a sphere.

The conditions for the existence of periodic solutions satisfying two types of commensurabilities are studied in detail: I) N'n' = 2n, II) N'n' = n, where n' is the mean orbital motion, n is the mean rotation rate of the sphere, and N' is a whole number. The main relationships de-

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scribed by the corresponding generating periodic solutions are formulated. The general and specific properties are noted for the cases of commensurabilities I) and II).

1. EQUATION OF MOTION

Let us consider the translational–rotational motion of a rigid body M_1 under the action of the gravitation of a uniform sphere M_0. We designate the masses of these bodies as m_1 and m_0 and the principal central moments of inertia of body M_1 as $A \neq B \neq C$.

Let Oxyz be a relative Cartesian coordinate system with the origin at the center of mass O of the sphere and with axes maintaining a constant orientation in space; O_1xyz is a coordinate system with the origin at the center of mass O_1 of body M_1 whose axes are parallel to the like axes of the coordinate system Oxyz; $O_1\xi\eta\zeta$ is a moving coordinate system whose axes are directed along the principal central axes of inertia of the body, with the principal central moments of inertia A, B, and C corresponding to the axis of inertia $O_1\xi$, $O_1\eta$, and $O_1\zeta$.

The relative translational–rotational motion of body M_1 is described by the Delaunay–Andoyer, canonical, osculating elements[5]

$$L', G', H', L, G, H, l', g', h', l, g, h, \tag{1}$$

where L', G', H', l', g', and h' are the Delaunay elements: $L' = \sqrt{\mu' a'}$, l' is the mean anomaly, $G' = \sqrt{\mu' a'(1-e'^2)}$, g' is the angular distance to the pericenter of the orbit, $H' = G'\cos i'$, h' is the longitude of the ascending node of the orbit, a' is the semimajor axis, e' is the eccentricity, i' is the inclination of the orbital plane to the principal Oxy, $\mu' = fm_0m_1\nu'$, $\nu' = m_0m_1/(m_0+m_1)$, and f is the gravitational constant; L, G, H, l, g, and h are the Andoyer elements: G is the magnitude of the vector **G** of the angular momentum of the rotational motion of the satellite, L is the projection of the vector **G** onto the $O_1\zeta$ axis of the body, H is the projection of the vector **G** onto the O_1z axis, l is the angle of the proper rotation of the satellite reckoned from the intermediate plane P normal to the vector **G**, g is the longitude of the ascending node of the plane $O_1\xi\eta$ of the body at the intermediate plane, and h is the longitude of the ascending node of the intermediate plane at the principal plane O_1xy. We also introduce into the analysis the quantities θ and ρ through the equations $L = G\cos\theta$ and $H = G\cos\rho$. Consequently, θ is the inclination of the plane $O_1\xi\eta$ of the body to the intermediate plane and ρ is the inclination of the plane P to the principal plane O_1xy.

In accordance with the general results of Ref. 5, the equations of motion of bodies M_0 and M_1 in the variables (1) have the form

$$\frac{d(L',G',H',L,G,H)}{dt}=\frac{\partial F}{\partial(l',g',h',l,g,h)},\quad \frac{\partial(l',g',h',l,g,h)}{dt}=-\frac{\partial F}{\partial(L',G',H',L,G,H)}, \tag{2}$$

where F is the characteristic function of the problem, expressed through the elements (1) using the equations of the unperturbed motion:

$$F=F(L', G', H', L, G, H, l', g', h-h', l, g). \tag{3}$$

Equations (2) admit of an energy integral and three area integrals:

$$F=c_1, \tag{4}$$
$$\sqrt{G'^2-H'^2}\sin h'+\sqrt{G^2-H^2}\sin h=c_2, \tag{5}$$
$$\sqrt{G'^2-H'^2}\cos h'+\sqrt{G^2-H^2}\cos h=c_3, \tag{6}$$
$$H'+H=c_4, \tag{7}$$

where c_1, c_2, c_3, and c_4 are arbitrary constants. The area integrals (5)-(7) are used to reduce the order of Eqs. (2). For this we take the fixed Laplace plane, for which $c_2 = c_3 = 0$ and $c_4 = c$, as the principal plane Oxy. Then Eqs. (5)-(7) are written in the form

$$H'={}^1\!/_2c+{}^1\!/_2c^{-1}(G'^2-G^2),\quad H={}^1\!/_2c-{}^1\!/_2c^{-1}(G'^2-G^2),\quad h-h'=\pi. \tag{8}$$

Equations (8) allow us to eliminate the variables h', h, H', and H from the right sides of Eqs. (2). For the variables G' and G contained in F, after this elimination, we introduce the new designations $G' = \Gamma'$ and $G = \Gamma$. Then (see Ref. 4) the equations for the variables L', Γ', L, Γ, l', g', l, and g form an independent system of equations which retains the canonical form

$$\frac{d(L',\Gamma',L,\Gamma)}{dt}=\frac{\partial K}{d(l',g',l,g)},\quad\frac{\partial(l',g',l,g)}{dt}=-\frac{\partial K}{\partial(L'\,\Gamma',L,\Gamma)}, \tag{9}$$

where K is the approximate value of the characteristic function of the problem, defined by the equation

$$K=K_0+\sigma K_1. \tag{10}$$

Here

$$K_0=\mu'^2/2\nu'L'^2-\Gamma^2/2A \tag{11}$$

is the principal part of the Hamiltonian corresponding to the chosen unperturbed motion in which the center of mass of body M_1 describes a Keplerian elliptical orbit while the body itself rotates as a uniform sphere with a constant angular velocity relative to an axis fixed in space; σK_1 are terms of first order relative to the small parameter σ, which are identified with the part of the kinetic energy of the rotational motion of the body due to the difference of this body from a sphere with a central moment of inertia A and with the second harmonic of the force function. In this case the remaining terms of the function K represent terms of higher order relative to the parameter σ, for which one can take the quantity $\sigma = \max\{|A-B|/A, |A-C|/A\}$ on the basis of the assumption that the ellipsoid of inertia of body M_1 is close to that of a sphere. We represent σK_1 as a function of the elements L', Γ', L, Γ, l', g', l, and g using the trigonometric expansion

$$\sigma K_1=-{}^1\!/_4(2/C-1/A-1/B)G^2\cos^2\theta-{}^1\!/_4(1/A-1/B)G^2\sin^2\theta\cos 2l +\lambda\Sigma U_{k_1,k_2,k_3,k_4}(\theta,J,e',\delta)\cos(k_1l'+k_2g'+k_3l+k_4g), \tag{12}$$

where the summation is carried out over the indices $k_1(0, \infty)$, $k_2(0, \pm 2)$, $k_3(0, \pm 2)$, and $k_4(0, \pm 1, \pm 2)$ while the coefficients $U_{k_1.k_2.k_3.k_4}$ as explicit functions of the variables

θ, J, and e' and the parameter δ are defined by the equations

$$
\begin{aligned}
&U_{s.0.0.0}={}^1/_2(1-{}^3/_2\sin^2\theta)(1-{}^3/_2\sin^2 J)(1-2\delta)C_s^{-3.0},\\
&U_{s.0.2\varepsilon.0}=-{}^3/_8\sin^2\theta(1-{}^3/_2\sin^2 J)C_s^{-3.0},\\
&U_{s.0.0.\nu}=-{}^3/_{16}\sin 2\theta\sin 2J(1-2\delta)C_s^{-3.0},\\
&U_{s.0.2\varepsilon.2\nu}=-{}^3/_{32}(1+\varepsilon\nu\cos\theta)^2\sin^2 JC_s^{-3.0},\\
&U_{s.0.0.2\nu}={}^3/_{16}\sin^2\theta\sin^2 J(1-2\delta)C_s^{-3.0},\\
&U_{s.0.2\varepsilon.\nu}=-{}^3/_{16}\varepsilon\nu\sin 2J\sin\theta(1+\varepsilon\nu\cos\theta)C_s^{-3.0},\\
&U_{s.2\mu.2\varepsilon.0}=-{}^9/_{32}\sin^2\theta\sin^2 J(C_s^{-3.2}+\mu S_s^{-3.2}),\\
&U_{s.2\mu.0.2\nu}={}^3/_{32}\sin^2\theta(1-\mu\nu\cos J)^2(1-2\delta)(C_s^{-3.2}+\mu S_s^{-3.2}),\\
&U_{s.2\mu.2\varepsilon.2\nu}=-{}^3/_{64}(1+\varepsilon\nu\cos\theta)^2(1-\mu\nu\cos J)^2(C_s^{-3.2}+\mu S_s^{-3.2}),\\
&U_{s.2\mu.0.\nu}={}^3/_{16}\sin J\sin 2\theta(\cos J-\mu\nu)(1-2\delta)(C_s^{-3.2}+\mu S_s^{-3.2}),\\
&U_{s.2\mu.2\varepsilon.\nu}={}^3/_{16}\varepsilon\nu\sin\theta\sin J(1+\varepsilon\nu\cos\theta)(\cos J-\mu\nu)(C_s^{-3.2}+\mu S_s^{-3.2}),\\
&U_{s.2\mu.0.0}={}^3/_8(1-{}^3/_2\sin^2\theta)\sin^2 J(1-2\delta)(C_3^{-3.2}+\mu S_s^{-3.2}),
\end{aligned}
\tag{13}
$$

where for brevity of writing we introduce the notation $\mu = \pm 1$, $\varepsilon = \pm 1$, and $\nu = \pm 1$ while s takes the values 0, 1, . . . , ∞;

$$\lambda={}^1/_2 fm_0(A-B)/a^3,\quad \delta=(A-C)/(A-B).$$

The coefficients $C_S^{-3.0}$, $C_S^{-3.2}$, and $S_S^{-3.2}$ are known functions of the eccentricity $e' = \sqrt{L'^2 - \Gamma'^2}/L'$, while the quantities θ, J, and a' are defined by the equations

$$
\begin{aligned}
&\cos\theta=L/\Gamma,\quad J=\rho+i',\\
&\cos J=(c^2-\Gamma'^2-\Gamma^2)/2\Gamma\Gamma',\ a'=L'^2/\mu'.
\end{aligned}
$$

Equations (9) admit of the integral K = const. After integration of Eqs. (9) the variables h', h, H', and H are calculated from Eqs. (8) using the quadrature

$$\dot{h}=-\int\frac{\partial F}{\partial H}dt+\text{const.}\tag{14}$$

Henceforth we will study periodic solutions of the equations which can be called solutions of the third kind in Poincaré's terminology. We note that these periodic solutions, generally speaking, will not correspond to periodic motions of bodies in space but will describe the periodic motions of bodies in a moving coordinate system rotating in the Laplace plane with an angular velocity $\dot{h} = -\partial F/\partial H$.

2. POINCARÉ PERIODIC SOLUTIONS OF THE THIRD KIND

According to Poincaré's theory,[1] the conditions for the existence of periodic solutions of Eqs. (9) have the form

$$
\begin{aligned}
&H(K_0)/_{L_0,\Gamma_0}\neq 0, && \text{A)}\\
&\partial[K_1]/\partial g_0'=\partial[K_1]/\partial l_0=\partial[K_1]/\partial g_0=0, && \text{B)}\\
&\partial[K_1]/\partial L_0=0, && \text{C)}\\
&\partial[K_1]/\partial\Gamma_0'=0, && \text{D)}\\
&H([K_1])/_{g_0',\,l_0,\,g_0,\,L_0,\,\Gamma_0'}\neq 0, && \text{E)}
\end{aligned}
$$

where L_0', L_0, Γ_0', Γ_0, l_0', l_0, g_0', and g_0 are arbitrary constants of the periodic generating solution of period T obtained as a result of the integration of Eqs. (9) with $\sigma = 0$:

$$
\begin{aligned}
&L'=L_0',\ \Gamma'=\Gamma_0',\ L=L_0,\ \Gamma=\Gamma_0,\\
&l'=n't+l_0',\quad g'=g_0',\quad l=l_0,\quad g=nt+g_0,\\
&n'=\mu'^2/\nu'L_0'^3,\quad n=\Gamma_0/A,\\
&k'n'=kn,\quad T=2\pi k'/n=2\pi k/n',
\end{aligned}
\tag{15}
$$

and k' and k are integers. In the conditions B)-E) $[K_1]$ is the Poincaré-averaged function K_1:

$$[K_1]=\frac{1}{T}\int_0^T K_1(L_0'\Gamma_0'L_0\Gamma_0 n't+l_0'g_0'l_0 nt+g_0)\,dt.\tag{16}$$

Since (16) is determined differently in the cases of commensurabilities I) and II), which are of practical interest in the given problem, we will study the conditions of existence A)-E) separately for these cases.

In case I), which corresponds to the actual motion of Mercury (N' = 3), the function $[K_1]$ is determined by the equation

$$
\begin{aligned}
\sigma[K_1]=&-{}^1/_4(2/C-1/A-1/B)G_0^2\cos^2\theta_0-{}^1/_4(1/B-1/A)\\
&\times G_0^2\sin^2\theta_0\cos 2l_0+\lambda[U_{0.0.0.0}+2U_{0.0.2.0}\cos 2l_0\\
&+U_{N'.0.0.-2}\cos(N'l_0'-2g_0)+U_{N'.0.2.-2}\cos(Nl_0'+2l_0-2g_0)\\
&+U_{N'.0.-2.-2}\cos(N'l_0'-2l_0-2g_0)+U_{N'.2.0.-2}\cos(Nl_0'\\
&+2g_0'-2g_0)+U_{N'.-2.0.-2}\cos(N'l_0'-2g_0'-2g_0)\\
&+U_{N'.2.2.-2}\cos(N'l_0'+2g_0'+2l_0-2g_0)+U_{N'.-2.2.-2}\cos(N'l_0'\\
&-2g_0'+2l_0-2g_0)+U_{N'.2.-2.-2}\cos(N'l_0'+2g_0'-2l_0-2g_0)\\
&+U_{N'.-2.-2.-2}\cos(N'l_0'-2g_0'-2l_0-2g_0)].
\end{aligned}
$$

Here the coefficients $U_{k_1.k_2.k_3.k_4}$ are calculated from Eqs. (13) with the generating values of the variables θ_0, J_0, e_0', and i_0' (henceforth we will omit the indices for brevity).

The condition A) is written the same way for both cases of commensurability and is clearly satisfied, since

$$H(K_0)|_{L_0,\Gamma_0}=\frac{3\mu'^2}{A\nu'L'^4}\neq 0$$

By solving Eqs. B) we find the generating values of the angular quantities:

$$
\begin{aligned}
&l_0'=0,\ g_0'=0,\ {}^1/_2\pi,\ \pi,\ {}^3/_2\pi;\ l_0=0,\ {}^1/_2\pi,\ \pi,\ {}^3/_2\pi;\\
&g_0=0,\ {}^1/_2\pi,\ \pi,\ {}^3/_2\pi.
\end{aligned}
\tag{17}
$$

The solution $g_0' = 0, {}^1/_2\pi, \pi, {}^3/_2\pi$ means that the direction toward the pericenter of the elliptical generating orbit either coincides with the line of intersection of the intermediate plane P with the orbital plane ($g_0' = 0, \pi$) or is orthogonal to this line ($g_0' = {}^1/_2\pi, {}^3/_2\pi$). The solution $l_0' = 0$; $l_0 = 0, {}^1/_2\pi, \pi, {}^3/_2\pi$; $g_0 = 0, {}^1/_2\pi, \pi, {}^3/_2\pi$ means that at the moment of passage through the pericenter of the orbit body M_1 occupies positions such that one of its axes of inertia either coincides with the line of apsides or is orthogonal to it.

With allowance for the generating values found for the angular variables (17), we can write the condition C) in explicit form. We will have

$$\cos\theta\Phi(J,\ \delta,\ e')=0,\tag{18}$$

where

$$\Phi(J,\delta,e')=\left(\frac{G^2}{3\lambda}\right)[(2/C-1/A-1/B)-\varepsilon_1(1/B-1/A)]$$
$$+(1-2\delta+\varepsilon_1)\{-(1-{}^3/_2\sin^2 J)C_0^{-3.0}+{}^1/_4\sin^2 JC_{N'}^{-3.0}\,\varepsilon_2$$
$$+{}^1/_4[C_{N'}^{-3.2}\,(1+\cos^2 J)+2\cos JS_{N'}^{-3.2}\,]\varepsilon_2\varepsilon_3\}.$$

Here we introduce the new notation $\varepsilon_1 = \cos 2l_0 = \pm 1$, $\varepsilon_2 = \cos 2g_0 = \pm 1$, and $\varepsilon_3 = \cos 2g_0' = \pm 1$. Equation (18) gives the generating solution $\theta = {}^1/_2\pi$.

The solution (17) and $\theta = {}^1/_2\pi$ means that one of the planes $O_1\xi\zeta$ or $O_1\eta\zeta$ of the body coincides with the intermediate plane P. The solution $l_0 = 0, {}^1/_2\pi, \pi, {}^3/_2\pi$; $\theta = {}^1/_2\pi$ means that the vector G is fixed in the body and coincides with one of the axes of inertia $O_1\eta$ or $O_1\zeta$ of the body.

The choice of the generating values of e', J, i', and δ is determined by the condition D), which is equivalent to three other equations:

$$\frac{1-e'^2}{e'}\frac{\partial[K_1]}{\partial e'}-\operatorname{ctg} i'\frac{\partial[K_1]}{\partial J}=0, \tag{19}$$
$$\sqrt{\mu'a'(1-e'^2)}\cos i'+\Gamma\cos\rho=c, \tag{20}$$
$$\sqrt{\mu'a'(1-e'^2)}\sin i'-\Gamma\sin\rho=0. \tag{21}$$

Equations (19)-(21) allow us to determine the generating values of i', ρ, and c as explicit functions of the generating values of e' and δ in a finite form (without resorting to the construction of solutions in series form). Here we assume that the quantities a' and Γ entering into (19)-(21) are given by the relation I, i.e.,

$$N'\mu'^2/\nu'L'^3=2\Gamma/A,$$

where N' is an odd number.

As the end result we will have

$$S_0+S_1c^2+S_2c^4=0, \tag{22}$$
$$i'=\arccos\left\{\frac{\mu'a'(1-e'^2)+c^2-\Gamma^2}{2\sqrt{\mu'a'(1-e'^2)}\,c}\right\},$$
$$\rho=\arccos\left\{\frac{-\mu'a'(1-e'^2)+c^2+\Gamma^2}{2\Gamma c}\right\}. \tag{23}$$

In Eq. (22) S_0, S_1, and S_2 are known functions of the quantities e', δ, ε_1, ε_2, and ε_3. After the generating values of c are calculated, the values of ρ and i' are determined by Eqs. (23), which follow directly from the area integrals (20) and (21). A numerical analysis of the solution (22), (23) allows us to establish the presence of solutions of the third kind. For this the final condition for existence is reduced to three others:

$$\Phi(J,e',\delta)\neq 0,\ \partial^2[K_1]/\partial\Gamma'^2\neq 0,\ H([K_1])_{g_0',\,l_0,\,g_0}\neq 0. \tag{24}$$

A study of the conditions (24) shows that they are satisfied for the generating solutions determined by (17), (18), (22), and (23).

Analogous arguments are easy to carry through for the case of the commensurability II), corresponding to the motion of synchronous satellites like the moon. This case has its specific properties, however.

The function $[K_1]$ is determined by the equation

$$\sigma[K_1]=-{}^1/_4(2/C-1/A-1/B)G_0^2\cos^2\theta_0-{}^1/_4(1/B-1/A)G_0^2$$
$$\times\sin^2\theta_0\cos 2l_0+\lambda\Sigma U_{sN',k_2,k_3,-s}\cos[s(Nl_0'-g_0)+k_2g_0'+k_3l_0],$$

where the summation indices take the values s(0, 1, 2), $k_2(0, \pm 2)$, and $k_3(0, \pm 2)$ while the coefficients $U_{sN'.k_2.k_3.-s}$ are calculated from Eqs. (13) with the generating values of the corresponding variables.

In this case Eqs. B) admit the solutions

$$l_0'=0;\ g_0'=0,\ {}^1/_2\pi,\ \pi,\ {}^3/_2\pi;\ l_0=0,\ {}^1/_2\pi,\ \pi,\ {}^3/_2\pi;\ g_0=0,\ \pi. \tag{25}$$

The solution $g_0' = 0, {}^1/_2\pi, \pi, {}^3/_2\pi$ means that in the generating solution the line of apsides either coincides with the line NN' of intersection of the intermediate plane with the orbital plane ($g_0' = 0, \pi$) or is orthogonal to this line ($g_0' = {}^1/_2\pi, {}^3/_2\pi$). The solution $g_0 = 0, \pi$ means that the equatorial plane of the body intersects the line NN'.

It is more convenient to write the conditions C) and D) relative to the generating values of the variables θ, i', ρ, J, and Γ together with the two area integrals and the condition of commensurability II). We will have the following system of equations:

$$1/\sin\theta\,\partial[K_1]/\partial\theta=0, \tag{26}$$
$$\frac{1-e'^2}{e'}\frac{\partial[K_1]}{\partial e'}-\operatorname{ctg} i'\frac{\partial[K_1]}{\partial J}=0, \tag{27}$$
$$\sqrt{\mu'a'(1-e'^2)}\cos i'+\Gamma\cos\rho=c, \tag{28}$$
$$\sqrt{\mu'a'(1-e'^2)}\sin i'-\Gamma\sin\rho=0, \tag{29}$$
$$N'\sqrt{\mu'}/\nu'a'^{3/2}=\Gamma/A,\quad J=i'+\rho, \tag{30}$$

where $[K_1]$, with allowance for the solutions (25), can be written in the form

$$\sigma[K_1]=\lambda[v]+\varepsilon_3\lambda[w],$$

where

$$[v]=v_0+\varepsilon_1v_1+\varepsilon_2v_2+\varepsilon_1\varepsilon_2v_{12},$$
$$[w]=w_0+\varepsilon_1w_1+\varepsilon_2w_2+\varepsilon_1\varepsilon_2w_{12}.$$

Here we introduce the notation $\varepsilon_1 = \cos 2l_0$, $\varepsilon_2 = \cos 2g_0'$, and $\varepsilon_3 = \cos g_0$, while the coefficients $v_0, v_1, \ldots, w_{12}$ are determined by the equations

$$v_0=\left[\frac{1}{2}\left(1-\frac{3}{2}\sin^2\theta\right)\left(1-\frac{3}{2}\sin^2 J\right)C_0^{-3.0}+\frac{3}{16}\sin^2\theta\sin^2 JC_{2N'}^{-3.0}\right]$$
$$\times(1-2\delta)-\frac{G^2}{4\lambda}\left(\frac{2}{C}-\frac{1}{A}-\frac{1}{B}\right)\cos^2\theta,$$
$$v_1=-\frac{3}{4}\sin^2\theta\left(1-\frac{3}{2}\sin^2 J\right)C_0^{-3.0}-\frac{3}{16}\left(1+\cos^2\theta\right)\sin^2 JC_{2N'}^{-3.0}$$
$$-\frac{G^2}{4\lambda}\left(\frac{1}{B}-\frac{1}{A}\right)\sin^2\theta,$$
$$v_2={}^3/_{16}\sin^2\theta(1-2\delta)[(1+\cos^2 J)C_{2N'}^{-3.2}+2\cos JS_{2N'}^{-3.2}\,],$$
$$v_{12}=-{}^3/_{16}(1+\cos^2\theta)[(1+\cos^2 J)C_{2N'}^{-3.2}-2\cos JS_{2N'}^{-3.2}\,],$$
$$w_0=-{}^3/_{16}\sin 2J\sin 2\theta(1-2\delta)C_{N'}^{-3.0},$$
$$w_1=-{}^3/_{16}\sin 2J\sin 2\theta C_{N'}^{-3.0},$$
$$w_2={}^3/_8\sin 2\theta\sin J(1-2\delta)(\cos JC_{N'}^{-3.2}+S_{N'}^{-3.2}),$$
$$w_{12}={}^3/_8\sin 2\theta\sin J(\cos JC_{N'}^{-3.2}+S_{N'}^{-3.2}).$$

Equations (26)-(30) cannot be solved rigorously in finite form relative to the generating values of c, i', ρ, and θ as explicit functions of e', δ, μ', a', Γ, and A like the way this was done in case I. Therefore, we must resort to some approximate method to solve Eqs. (26)-(30).

Let us indicate one possible means of solving Eqs. (26)-(30). For a number of celestial bodies whose mean motions n' and n satisfy the condition of commensurability II) the quantities ρ, i', and e' can have small values (in the case of the moon, for example). Then the function [w] has the order of smallness Je' in comparison with the function [v]. Neglecting terms of a higher order of smallness, we arrive at the equations of the zeroth approximation:

$$\frac{1}{\sin\theta_0}\frac{\partial[v]}{\partial\theta_0}=0, \tag{31}$$

$$\frac{1-e_0'^2}{e_0'}\frac{\partial[v]}{\partial e_0'}-\operatorname{ctg} i_0'\frac{\partial[v]}{\partial J_0}=0, \tag{32}$$

$$\sqrt{\mu' a_0'(1-e_0'^2)}\cos i_0'+\Gamma_0\cos\rho_0=c_0, \tag{33}$$

$$\sqrt{\mu' a_0'(1-e_0'^2)}\sin i_0'-\Gamma_0\sin\rho_0=0. \tag{34}$$

It is easy to show that Eq. (31) has the single solution $\theta_0 = {}^1/_2\pi$, while Eqs. (32)-(34) can be solved in finite form for c_0, i_0', and ρ_0 by the method presented above.

Then an exact solution of Eqs. (26)-(30) is constructed in the form of series for which the solution of Eqs. (31)-(34) serves as the zeroth approximation.

The conclusion follows from what was said above that for periodic solutions in the case of commensurability II) the vector G of angular momentum of the rotational motion of body M_1 does not coincide with any axis of inertia. In this case Eqs. (26)-(30) allow one to give a theoretical estimate of this departure in the generating solution.

CONCLUSION

The existence of Poincaré periodic solutions of the third kind in the problem of the translational–rotational motion of a rigid body in the gravitational field of a sphere is demonstrated in the report. These solutions describe the periodic motions of bodies in a moving coordinate system with the origin at the center of the sphere and rotating in the Laplace plane together with the line of nodes. In this case the law of motion of a node is determined by the quadrature (14) and in the general case it does not correspond to periodic motions of bodies in space.

In the corresponding generating solutions the arbitrary quantities are: the semimajor axis a', the eccentricity e', and the initial value of the mean anomaly l_0', as well as the numerical values of the constant parameters of the problem: the mass m_i of one of the bodies, the moment of inertia A of the body, and the dynamic parameter δ = (A − C)/(A − B).

The generating solutions correspond to unperturbed translational–rotational motions of the bodies described by the following relationships:

1. The descending node of the intermediate plane coincides with the ascending node of the orbit in the Laplace plane (this situation remains valid for any exact solution of Eq. (2)].

2. One of the planes $O_1\xi\zeta$ or $O_1\eta\zeta$ of the body (we call it the equatorial plane of the body) intersects the Laplace plane along the line of nodes NN' of the orbit or along a line orthogonal to it at the moment the center of mass of the body M_1 passes through the pericenter of the orbit.

3. The line of apsides either coincides with the line of nodes or is orthogonal to it.

4. The vector G of angular momentum is fixed in the body M_1 and coincides with one of the axis of inertia $O_1\xi$ or $O_1\eta$ (we call it the polar axis) in the case of commensurability I) and is distant from the polar axis by an angle ${}^1/_2\pi - \theta$ in the case of commensurability II). In the second case the vector G lies in one of the planes $O_1\xi\eta$ or $O_1\eta\zeta$ of the body while the constant value of θ is determined as a result of the solution of Eqs. (26)-(30).

5. The vector G is fixed in space and forms a constant angle ρ with the fixed Laplace plane and an angle ρ + i' with the fixed orbital plane. The generating values of the quantities ρ and i' are determined through the solution of Eqs. (19)-(21) (case I) and (26)-(30) (case II).

6. At the moment the center of mass of body M_1 passes through the pericenter of the orbit one of its principal central axes either coincides with the line of apsides or is orthogonal to it and lies in the orbital plane.

For the first case of commensurability finite equations were obtained for finding the generating values of i', ρ, and J from the given a', e', δ, m_0, A, ε_1, ε_2, and ε_3, which makes them suitable for practical applications. In the other case of commensurability of the mean motions an approximate means of finding the generating values of θ, i', ρ, and J is indicated.

An important property of periodic solutions of type II is that the vector G does not coincide with the polar axis of inertia of the body (as occurs, for example, in the simpler model of Ref. 6 in accordance with Cassini's laws). In this case Eqs. (26)-(30) allow one to make a theoretical estimate of the indicated effect.

We note that in the present work we were confined to an approximate value of the characteristic function of the problem, retaining the main expansion terms up to the second harmonic inclusively in the expansion of the force function. For a wide class of rigid bodies M_1, however, with the proper introduction of the small parameter the results of the work will be valid for an exact value of the force function of the Newtonian interaction of bodies M_0 and M_1.

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