STABILITY OF POSSIBLY NONISOLATED SOLUTIONS OF CONSTRAINED EQUATIONS, WITH APPLICATIONS TO COMPLEMENTARITY AND EQUILIBRIUM PROBLEMS*

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January 31, 2017 (Last revised October 19, 2017)

ABSTRACT

We present a new covering theorem for a nonlinear mapping on a convex cone, under the assumptions weaker than the classical Robinson's regularity condition. When the latter is violated, one cannot expect to cover the entire neighborhood of zero in the image space. Nevertheless, our covering theorem gives rise to natural conditions guaranteeing stability of a solution of a cone-constrained equation subject to wide classes of perturbations, and allowing for nonisolated solutions, and for systems with the same number of equations and variables. These features make these results applicable to various classes of variational problems, like nonlinear complementarity problems. We also consider the related stability issues for generalized Nash equilibrium problems.

Key words: constrained equation; singular solution; nonisolated solution; covering; stability; sensitivity; complementarity problem; generalized Nash equilibrium problem.
AMS subject classifications. 47J05, 49J53, 90C33.

Dedicated to the memory of Professor Jonathan Borwein.

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* Research of the first author is supported in part by the Russian Foundation for Basic Research Grant 17-01-00125, by the Ministry of Education and Science of the Russian Federation (Project 1.962.2017/4.6), by the RUDN University Program 5-100, and by the Volkswagen Foundation. The second author is supported by the Russian Science Foundation Grant 17-11-01168.

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1 Introduction

This paper is concerned with stability properties of a given solution $\bar{u}$ of constrained equation

$$\Phi(u) = 0, \quad u \in P, \tag{1.1}$$

where $\Phi : \mathbb{R}^p \to \mathbb{R}^q$ is a sufficiently smooth mapping (the exact smoothness requirements will be specified as needed), and $P \subset \mathbb{R}^p$ is a closed convex set. In the main results presented in Section 2 it will be assumed that the set P is conical at the solution $\bar{u}$ in question, which means that the set $P - \bar{u}$ behaves near zero like a cone (the formal definition will be given below). This setting includes the case when P is polyhedral, allowing us to cover important applications in Sections 3 and 4. That said, of course, the conicity assumption reduces the area of applicability of our results; extending them to possibly non-conical P is the subject of our ongoing research.

We are mostly interested in those cases when the solution in question can be in some sense singular, and in particular, is not necessarily isolated. Specifically, we wish to obtain conditions ensuring stability of a given solution subject to wide classes of perturbations, despite the fact that every neighborhood of this solution may contain other solutions. In this paper we restrict ourselves to the case of right-hand side perturbations, i.e., we consider the parametric family of problems

$$\Phi(u) = w, \quad u \in P, \tag{1.2}$$

where $w \in \mathbb{R}^q$ is a perturbation parameter.

For unconstrained equation

$$\Phi(u) = 0 \tag{1.3}$$

(i.e., when $P = \mathbb{R}^p$), these issues have been studied in [22]. In this case, the meaning of singularity is clear: a solution $\bar{u}$ of (1.3) is singular if it violates the regularity condition

$$\operatorname{rank} \Phi'(\bar{u}) = q,$$

the latter being equivalent to

$$\operatorname{im} \Phi'(\bar{u}) = \mathbb{R}^q, \tag{1.4}$$

where im stands for the range space of a linear operator. If $\bar{u}$ is a nonisolated solution of (1.3), then the contingent cone $T_{\Phi^{-1}(0)}(\bar{u})$ to the solution set $\Phi^{-1}(0)$ at $\bar{u}$ is nontrivial, and hence, $\ker \Phi'(\bar{u})$ containing $T_{\Phi^{-1}(0)}(\bar{u})$ is also nontrivial, where ker stands for the null space of a linear operator. In particular, if $p = q$, then $\Phi'(\bar{u})$ is a singular square matrix, and hence, $\bar{u}$ is necessarily a singular solution.

In [22, Theorem 5], it was shown that a solution $\bar{u}$ of (1.3) "survives" perturbations in large classes if Φ is smooth enough, and there exists $\bar{v} \in \ker \Phi'(\bar{u})$ such that Φ is 2-regular at $\bar{u}$ in the direction $\bar{v}$, the latter meaning that

$$\operatorname{im} \Phi'(\bar{u}) + \Phi''(\bar{u})[\bar{v}, \ker \Phi'(\bar{u})] = \mathbb{R}^q. \tag{1.5}$$

This notion is a useful tool in nonlinear analysis and optimization theory; see, e.g., the book [1], and [17, 18] for some recent applications. Evidently, (1.5) holds automatically with every

$\bar{v} \in \mathbb{R}^p$ (including $\bar{v} = 0$) provided the regularity condition (1.4) is satisfied. At the same time, (1.5) may hold with nonzero $\bar{v}$ even if (1.4) is violated, and even if $\bar{u}$ is a nonisolated solution; see the examples in [22] and below.

Furthermore, as demonstrated in [22], condition (1.5) may never hold with $\bar{v} \in \ker \Phi'(\bar{u})$ if $p = q$ (which will be the case of special interest in this work), $\bar{u}$ is a singular solution, and $T_{\Phi^{-1}(0)}(\bar{u}) = \ker \Phi'(\bar{u})$, the latter being one of the two ingredients of the concept of noncriticality of solution $\bar{u}$, as introduced in [22]. The second ingredient is Clarke regularity of $\Phi^{-1}(0)$ at $\bar{u}$, and as demonstrated in [22, Theorem 1], under the appropriate smoothness assumptions, this combination of properties is equivalent to the local Lipschitzian error bound

$$\operatorname{dist}(u,\, \Phi^{-1}(0)) = O(\|\Phi(u)\|) \quad \text{as } u \in \mathbb{R}^p \text{ tends to } \bar{u}, \tag{1.6}$$

which is known to be equivalent to the following upper Lipschitzian property, the concept dating back to [31]:

$$\operatorname{dist}(u(w),\, \Phi^{-1}(0)) = O(\|w\|) \quad \text{as } w \in \mathbb{R}^q \text{ tends to } 0,$$

where $u(w)$ is any solution of the perturbed equation

$$\Phi(u) = w, \tag{1.7}$$

close enough to $\bar{u}$. In addition, [22, Proposition 1] implies that singular noncritical solutions of (1.3) can only be stable subject to very special perturbations. At the same time, critical solutions (i.e., those which are not noncritical), or, more precisely, those solutions for which $T_{\Phi^{-1}(0)}(\bar{u})$ is a proper subset of $\ker \Phi'(\bar{u})$, can naturally satisfy (1.5) with some $\bar{v} \in \ker \Phi'(\bar{u})$, even if $p = q$, and hence, be stable subject to wide classes of perturbations.

Our goal here is to investigate the possibilities of (at least partial) extension of these considerations to constrained equations of the form (1.1), which is a very rich problem setting encompassing a much wider area of applications than (1.3).

Let $\mathcal{S}$ stand for the solution set of (1.1). The results for unconstrained equations, outlined above, might give rise to a conjecture that the constrained local Lipschitzian error bound property at $\bar{u} \in S$, which consists of saying that

$$\operatorname{dist}(u,\, \mathcal{S}) = O(\|\Phi(u)\|) \quad \text{as } u \in P \text{ tends to } \bar{u}, \tag{1.8}$$

is equivalent to the combination of the equality $T_{\mathcal{S}}(\bar{u}) = \ker \Phi'(\bar{u}) \cap T_P(\bar{u})$ and Clarke regularity of $\mathcal{S}$ at $\bar{u}$, perhaps under some additional requirements regarding P. However, this is not true, in general, even when P is a half-space (i.e., is defined by a single linear constraint), as demonstrated by the following example.

Example 1.1 Take the union U of two closed Euclidian balls in $\mathbb{R}^p$ ($p \geq 2$) with the only common point $\bar{u} = 0$, and an infinitely differentiable function $\Phi : \mathbb{R}^p \to \mathbb{R}$ such that $U = \Phi^{-1}(0)$ (such function exists, due to Whitney's theorem [6, Theorem 2.3.1]). Let $a \in \mathbb{R}^p \setminus \{0\}$ be such that the hyperplane $\{u \in \mathbb{R}^p \mid \langle a,\, u\rangle = 0\}$ separates the specified two balls. If we set $P = \{u \in \mathbb{R}^p \mid \langle a,\, u\rangle \leq 0\}$, then $\mathcal{S}$ coincides with one of the balls and hence, is Clarke-regular at every its point; see Figure 1. Moreover, $\Phi'(\bar{u}) = 0$, and hence, $\ker \Phi'(\bar{u}) \cap T_P(\bar{u}) = T_{\mathcal{S}}(\bar{u}) =$

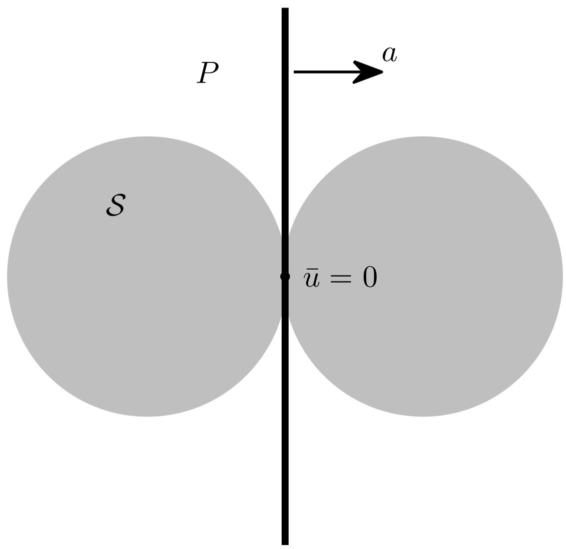

Figure 1: Solution set from Example 1.1.

P. However, from [22, Example 2] it follows that the unconstrained error bound cannot hold at $\bar{u}$ for any appropriate Φ, evidently implying that the constrained error bound also cannot hold.

Nevertheless, even in the absence of a "verifiable" equivalent characterization of the constrained error bound (1.8), the latter can itself be regarded as the property defining noncritical solutions. The idea of doing so had been emphasized in the discussion associated to [26] (see [14, 27]). It can be easily checked that similarly to the unconstrained case, the constrained error bound (1.8) is equivalent to the upper Lipschitzian property which consists of saying that

$$\operatorname{dist}(u(w), \mathcal{S}) = O(\|w\|) \quad \text{as } w \in \mathbb{R}^q \text{ tends to } 0,$$

where $u(w)$ is any solution of the perturbed constrained equation (1.2), close enough to $\bar{u}$.

However, further pursuing the line of development for the unconstrained case is not at all straightforward. To begin with, it is in general not evident what should be regarded as singularity of a solution $\bar{u}$ of (1.1). A seemingly natural understanding of singularity might be violation of Robinson's regularity condition

$$0 \in \operatorname{int} \Phi'(\bar{u})(P - \bar{u}). \tag{1.9}$$

However, if $p = q$, then (1.9) can only hold when $\bar{u} \in \operatorname{int} P$, in which case any local analysis would be essentially concerned with the unconstrained equation (1.3). In other words, any solution $\bar{u} \notin \operatorname{int} P$ would automatically be regarded as singular, and such understanding of singularity would be unreasonably weak.

Furthermore, if $\bar{u}$ is a nonisolated solution of (1.1), then $\ker \Phi'(\bar{u})$ (and even $\ker \Phi'(\bar{u}) \cap T_P(\bar{u})$) is necessarily nontrivial. This suggests to keep considering violation of (1.4) (which is of course stronger than violation of (1.9)) as a possible understanding of singularity, at least when $p = q$, in which case such singularity of $\bar{u}$ is equivalent to saying that $\Phi'(\bar{u})$ is a singular square matrix.

Repeating with evident modifications the argument in [22, Proposition 1], we come to the following conclusion. Assuming that Φ is smooth enough, and the constrained error bound (1.8) holds at $\bar{u} \in \mathcal{S}$, consider any sequences $\{w^k\} \subset \mathbb{R}^q \setminus \{0\}$, $\{u^k\} \subset \mathbb{R}^p$, and $\{\hat{u}^k\}$, such that $\{w^k\} \to 0$, $\{u^k\} \to \bar{u}$, and for each k it holds that u^k is a solution of (1.2) with $w = w^k$, while $\hat{u}^k$ is some projection of u^k onto $\mathcal{S}$. Then the sequence $\{(w^k,\, u^k - \hat{u}^k)/\|w^k\|\}$ is bounded (because of the upper Lipschitzian property equivalent to the constrained error bound), and any accumulation point $(d,\, v)$ of this sequence satisfies the equality $\Phi'(\bar{u})v = d$. The latter implies the inclusion $d \in \operatorname{im} \Phi'(\bar{u})$, where the right-hand side is a proper linear subspace in $\mathbb{R}^q$ provided (1.4) is violated. Therefore, a singular solution satisfying the constrained local Lipschitzian error bound can only be stable subject to very special right-hand side perturbations, i.e., those tangential to $\operatorname{im} \Phi'(\bar{u})$.

In Section 2, we will establish a new covering result for a mapping on a cone, under the assumptions weaker than Robinson's regularity, and even allowing for singularity of a solution in question in the sense of violation of (1.4). In these cases, one cannot expect to cover the entire neighborhood of 0 in $\mathbb{R}^q$, but the set being covered can be guaranteed to be "large" under some additional conditions. In the case when $p = q$ and the solution in question is singular, the corresponding additional condition can never be satisfied if the constrained local Lipschitzian error bound holds at this solution, the observation agreeing with the discussion above. At the same time, if the error bound is violated, this condition may naturally hold.

Furthermore, in Sections 3 and 4, we will consider applications of the results obtained in the context of complementarity problems and generalized Nash equilibrium problems.

Some words about our notation, which is fairly standard. Let $B(u,\, t)$ ($S(u,\, t)$) stand for the closed ball (sphere) in a metric space U, centered at $u \in U$, and with radius $t > 0$. By $\operatorname{span} U$, $\operatorname{cone} U$, and $\operatorname{ri} U$,we denote the linear space spanned by $U \subset \mathbb{R}^p$, the conic hull of U, and the relative interior of U, respectively. The identity operator will be denoted by I. For a vector u, let u_J stand for the subvector with components u_j, $j \in J$. Similarly, for a matrix M, let $M_{J_1 J_2}$ stand for the submatrix with rows numbered by $j_1 \in J_1$ and columns numbered by $j_2 \in J_2$. We write $|J|$ for the cardinality of a finite set J.

2 Covering results

In this section we will consider problem (1.1) under the additional assumption that P is conical at the solution $\bar{u}$ in question, by which we mean that near 0, the radial cone $R_P(\bar{u}) = \operatorname{cone}(P - \bar{u})$ to P at $\bar{u}$ coincides with $P - \bar{u}$. In this case, in local considerations one can replace P in (1.1) by $\bar{u} + R_P(\bar{u})$. For this reason, we restrict ourselves to the following problem setting:

$$\Phi(u) = 0, \quad u \in \bar{u} + K,$$

where $\Phi : \mathbb{R}^p \to \mathbb{R}^q$ is a twice differentiable mapping, $K \subset \mathbb{R}^p$ is a closed convex cone, and $\bar{u} \in \Phi^{-1}(0)$.

We start with the important special case when Φ is affine.

Theorem 2.1 *Let $\Phi(u) = Au + b$ with some linear operator $A : \mathbb{R}^p \to \mathbb{R}^q$ and $b \in \mathbb{R}^q$, and let $\bar{u} \in \Phi^{-1}(0)$.*

Then for any closed convex cone $C \subset \mathbb{R}^q$ satisfying $C \setminus \{0\} \subset \mathrm{ri}\, AK$, there exists $\theta > 0$ such that for every $w \in C$, there exists $u(w) \in \bar{u} + K$ such that $\Phi(u(w)) = w$, and $\|u(w) - \bar{u}\| \leq \|w\|/\theta$.

Moreover, if K is polyhedral, the assertion above holds for $C = AK$.

This result can be regarded as a generalization of the (finite-dimensional) Banach open mapping theorem to linear operators on cones. The main assertion follows immediately from [2, Corollary 3], while the proof of the last assertion can be found in [2, p. 448] (this result is also mentioned in [29, Theorem 2]). If K is not polyhedral, one cannot take $C = AK$, in general, even assuming that AK is closed; the following counterexample was proposed in [35].

Example 2.1 Define the closed convex cone $C = \{w \in \mathbb{R}^3 \mid (w_1 - w_3)^2 + w_2^2 \leq w_3^2,\ w_3 \geq 0\}$. Let $\tilde{B}$ be the intersection of C with the cylinder $\{w \in \mathbb{R}^3 \mid (w_1 - 1)^2 + w_3^2 \leq 1\}$, and let $B = \mathrm{conv}(\tilde{B} \cup \{(0,\, 0,\, 1)\})$.

Now we embed the instance of $\mathbb{R}^3$, containing the defined objects, into $\mathbb{R}^4$, and fix $e^0 \in \mathbb{R}^4 \setminus \{0\}$ such that it is orthogonal to the specified instance of $\mathbb{R}^3$. Set $K = \mathrm{cone}(B + e^0)$, with B considered as a subset of $\mathbb{R}^4$, and let A be the orthogonal projector in $\mathbb{R}^4$ onto $\mathbb{R}^3$. Evidently, for every $w \in C$ there exists $t > 0$ such that $tw \in B$. Therefore, $AK = C$, while $A(\cup_{t \in [0,\, 1]}(B + e^0)) = B$. It remains to observe that one can approach 0 staying in C but beyond B.

The proof of Theorem 2.2 below requires some formal concepts of covering and the related stability result which we present next.

A mapping $\Psi : U \to W$ between metric spaces U and W is said to be covering at a linear rate with a constant $\theta > 0$ with respect to a set $V \subset U$ if

$$B(\Psi(u),\, \theta t) \subset \Psi(B(u,\, t)) \quad \forall\, u \in U,\ \forall\, t \geq 0 \text{ such that } B(u,\, t) \subset V.$$

If Ψ is covering at a linear rate with a constant θ with respect to $V = U$, we will be simply saying that Ψ is covering at a linear rate with this constant, which amounts to the property

$$B(\Psi(u),\, \theta t) \subset \Psi(B(u,\, t)) \quad \forall\, u \in U,\ \forall\, t \geq 0,$$

and which evidently implies covering at a linear rate with the same constant with respect to every $V \subset U$. When there will be no need to specify the constant of covering, we will be saying that Ψ is covering at a linear rate (assuming that this holds with some constant). The study of covering properties in metric spaces dates back to [9] at least.

One useful observation is that if Ψ is a restriction of linear operator $A : \mathbb{R}^p \to \mathbb{R}^q$ to a convex cone $U = K \subset \mathbb{R}^p$ (with $W = \mathbb{R}^q$), then covering at a linear rate with a constant θ is in fact equivalent to the following covering property: for every $w \in \mathbb{R}^q$, there exists $u(w) \in K$ such that $Au(w) = w$, and $\|u(w)\| \leq \|w\|/\theta$. Furthermore, according to Theorem 2.1 applied with $b = 0$ and $C = \mathbb{R}^q$, the latter is equivalent to the equality $AK = \mathbb{R}^q$ (this can also be easily verified directly).

The following is a corollary of a more general result derived in [4, 5].

Proposition 2.1 *Let U and W be metric spaces, let U be complete, and let $\Psi : U \to W$ be a continuous mapping. Let Ψ be covering at a linear rate with a constant $\theta > 0$ with respect to $B(u^0,\, \rho)$ for some $u^0 \in U$ and $\rho > 0$. Let a mapping $\Omega : U \to W$ by Lipschitz-continuous on $B(u^0,\, \rho)$ with a constant $\ell \in (0,\, \theta)$, and assume that*

$$\|\Psi(u^0) - \Omega(u^0)\| < (\theta - \ell)\rho.$$

Then there exists $u \in B(u^0,\, \rho)$ such that $\Psi(u) = \Omega(u)$.

We are now in a position to prove the main result of this paper.

Theorem 2.2 *Let Φ be twice differentiable near $\bar{u} \in \Phi^{-1}(0)$, with its second derivative being continuous at $\bar{u}$, and let Φ be 2-regular at $\bar{u}$ with respect to K in a direction $\bar{v} \in K$, i.e.,*

$$\operatorname{span} \Phi'(\bar{u}) R_K(\bar{v}) + \Phi''(\bar{u})[\bar{v},\, \ker \Phi'(\bar{u}) \cap R_K(\bar{v})] = \mathbb{R}^q. \tag{2.1}$$

Let $\|\bar{v}\| < 1$.

Then for any closed convex cone $C \subset \mathbb{R}^q$ satisfying $C \setminus \{0\} \subset \operatorname{ri} \Phi'(\bar{u})K$, any $\bar{w} \in \operatorname{ri} \Phi'(\bar{u})K$, and any $\varepsilon > 0$, there exist $\theta > 0$ and $t_0 > 0$ such that for all $t \in [0,\, t_0]$ it holds that

$$\varphi_{\bar{v},\, \bar{w}}(t) + \Gamma_{C,\, \theta}(t) \subset \Phi((\bar{u} + K \cap \operatorname{cone} B(\bar{v},\, \varepsilon)) \cap B(\bar{u},\, t)), \tag{2.2}$$

where

$$\varphi_{\bar{v},\, \bar{w}}(t) = t\Phi'(\bar{u})\bar{v} + t^2 \left(\frac{1}{2} \Phi''(\bar{u})[\bar{v},\, \bar{v}] + \bar{w} \right),$$

$$\Gamma_{C,\, \theta}(t) = B(0,\, \theta t) \cap C + B(0,\, \theta t^2).$$

Moreover, if K is polyhedral, the assertion above holds for $C = \Phi'(\bar{u})K$.

Proof. In the argument below, we cannot use condition (2.1) as it is, because the radial cone $R_K(\bar{v})$ is not necessarily closed. To that end, we next show that (2.1) still holds if we replace $R_K(\bar{v})$ in it by some polyhedral (hence, closed and convex) cones contained in $R_K(\bar{v})$.

Indeed, let $v_1^1,\, \ldots,\, v_r^1$ be any nonzero vectors in $R_K(\bar{v})$ such that $\operatorname{span}\{\Phi'(\bar{u})v_1^1,\, \ldots,\, \Phi'(\bar{u})v_r^1\} = \operatorname{span} \Phi'(\bar{u}) R_K(\bar{v})$ and $\bar{w} \in \operatorname{ri} \operatorname{cone}\{\Phi'(\bar{u})v_1^1,\, \ldots,\, \Phi'(\bar{u})v_r^1\}$. Furthermore, let $w_1^2,\, \ldots,\, w_s^2 \in \mathbb{R}^q$ be any nonzero vectors such that

$$\operatorname{span} \Phi'(\bar{u}) R_K(\bar{v}) + \operatorname{cone}\{w_1^2,\, \ldots,\, w_s^2\} = \mathbb{R}^q. \tag{2.3}$$

Condition (2.1) implies that for every $j \in \{1, \ldots, s\}$, there exists (necessarily nonzero) $v_j^2 \in \ker \Phi'(\bar{u}) \cap R_K(\bar{v})$ such that

$$\Phi''(\bar{u})[\bar{v},\, v_j^2] - w_j^2 \in \operatorname{span} \Phi'(\bar{u}) R_K(\bar{v}). \tag{2.4}$$

Set $K_1 = \operatorname{cone}\{v_1^1, \ldots, v_r^1\} \subset R_K(\bar{v})$, $K_2 = \operatorname{cone}\{v_1^2, \ldots, v_s^2\} \subset \ker \Phi'(\bar{u}) \cap R_K(\bar{v})$. Evidently,

$$\operatorname{span} \Phi'(\bar{u}) R_K(\bar{v}) = \operatorname{span} \Phi'(\bar{u}) K_1. \tag{2.5}$$

According to (2.3), any vector $w \in \mathbb{R}^q$ can be written as $w = w^1 + \sum_{j=1}^{s} \beta_j w_j^2$ with some $w^1 \in \operatorname{span} \Phi'(\bar{u}) R_K(\bar{v})$ and $\beta_j \geq 0$, $j \in \{1, \ldots, s\}$, and hence, for $v^2 = \sum_{j=1}^{s} \beta_j v_j^2 \in K_2$ it holds that

$$\begin{aligned}
\Phi''(\bar{u})[\bar{v},\, v^2] - w &= \Phi''(\bar{u}) \left[\bar{v},\, \sum_{j=1}^{s} \beta_j v_j^2 \right] - w^1 - \sum_{j=1}^{s} \beta_j w_j^2 \\
&= \sum_{j=1}^{s} \beta_i \left(\Phi''(\bar{u})[\bar{v},\, v_j^2] - w_j^2 \right) - w^1 \\
&\in \operatorname{span} \Phi'(\bar{u}) R_K(\bar{v}) \\
&= \operatorname{span} \Phi'(\bar{u}) K_1,
\end{aligned}$$

where the inclusion is by (2.4), and the last equality is by (2.4). This proves that

$$\operatorname{span} \Phi'(\bar{u}) K_1 + \Phi''(\bar{u})[\bar{v},\, K_2] = \mathbb{R}^q. \tag{2.6}$$

Let for convenience $\|v_i^1\| = 1$ for all $i \in \{1, \ldots, r\}$, and $\|v_j^2\| = 1$ for all $j \in \{1, \ldots, s\}$. Since $v_i^1 \in R_K(\bar{v})$, there exists $\tau_1 > 0$ such that $\bar{v} + \tau_1 v_i^1 \in K$ for all $i \in \{1, \ldots, r\}$. Any $v^1 \in K_1$ can be written as $v^1 = \sum_{i=1}^{r} \alpha_i v_i^1$ with some $\alpha_i \geq 0$, $i \in \{1, \ldots, r\}$, which can be chosen in such a way that the vectors v_i^1 with $\alpha_i > 0$ are linearly independent (see, e.g., [33, Corollary 17.1.2]). Since the number of linearly independent subsystems of $v_1^1, \ldots, v_r^1$ is finite, it can be easily seen that there exists $c_1 > 0$ such that for every $v^1 \in K_1$, one can chose α_i, $i \in \{1, \ldots, r\}$, in such a way that $\alpha \leq c_1 \|v^1\|$, where $\alpha = \sum_{i=1}^{r} \alpha_i$.

Similarly, there exists $\tau_2 > 0$ such that $\bar{v} + \tau_2 v_j^2 \in K$ for all $j \in \{1, \ldots, s\}$, and there exists $c_2 > 0$ such that every $v^2 \in K_2$ can be written as $v^1 = \sum_{j=1}^{r} \beta_j v_j^2$ with some $\beta_j \geq 0$, $j \in \{1, \ldots, s\}$, such that $\beta \leq c_2 \|v^2\|$, where $\beta = \sum_{j=1}^{s} \beta_j$.

Therefore, assuming that $v^1 \neq 0$ or $v^2 \neq 0$ (and hence, $\alpha + \beta > 0$), and employing convexity of K, we derive

$$\begin{aligned}
\bar{v} + v^1 + v^2 &= \bar{v} + (\alpha + \beta) \sum_{i=1}^{r} \frac{\alpha_i}{\alpha + \beta} v_i^1 + (\alpha + \beta) \sum_{j=1}^{s} \frac{\beta_j}{\alpha + \beta} v_j^2 \\
&= \sum_{i=1}^{r} \frac{\alpha_i}{\alpha + \beta} \left(\bar{v} + (\alpha + \beta) v_i^1 \right) + \sum_{j=1}^{s} \frac{\beta_j}{\alpha + \beta} \left(\bar{v} + (\alpha + \beta) v_j^2 \right) \\
&\in K
\end{aligned}$$

provided $\alpha + \beta \leq \min\{\tau_1,\, \tau_2\}$. Taking into account that

$$\alpha + \beta \leq c_1 \|v^1\| + c_2 \|v^2\| \leq (c_1 + c_2) \max\{\|v^1\|,\, \|v^2\|\},$$

the needed property necessarily holds if $\max\{\|v^1\|,\, \|v^2\|\} \leq \tau$, where $\tau = \min\{\tau_1,\, \tau_2\}/(c_1 + c_2)$.

We thus proved that $\bar{v} + v^1 + v^2 \in K$ for all $v^1 \in K_1 \cap B(0,\, \tau)$ and $v^2 \in K_2 \cap B(0,\, \tau)$.

Furthermore, according to Theorem 2.1, there exists $\theta_1 > 0$ such that for every $w^1 \in C$, there exists $v(w^1) \in K$ satisfying $\Phi'(\bar{u})v(w^1) = w^1$ and $\|v(w^1)\| \leq \|w^1\|/\theta_1$.

For given $t > 0$, $w^1 \in C$, and $w^2 \in \mathbb{R}^q$, we need to find a solution of the equation

$$\Phi(u) = t(\Phi'(\bar{u})\bar{v} + w^1) + t^2 \left(\frac{1}{2} \Phi''(\bar{u})[\bar{v},\, \bar{v}] + \bar{w} + w^2 \right). \tag{2.7}$$

We will construct such solution in the form $u = \bar{u} + tu(t,\, w^1,\, v^1,\, v^2)$, where $u(t,\, w^1,\, v^1,\, v^2) = \bar{v} + v(w^1) + tv^1 + v^2$, $v^1 \in K_1$, $v^2 \in K_2$. Observe that according to the conclusions obtained above, if $t\|v^1\| \leq \tau$ and $\|v^2\| \leq \tau$, then $u(t,\, w^1,\, v^1,\, v^2) \in K$.

Fix any $v^0 \in K$ such that $\Phi'(\bar{u})v^0 = \bar{w}$. Since $\|\bar{v}\| < 1$, there exist $t_0 > 0$, $\theta > 0$, and $\rho > 0$, such that

$$t_0(\|v^0\| + \rho) \leq \tau, \quad \rho \leq \tau, \quad \theta/\theta_1 + t_0(\|v^0\| + \rho) + \rho \leq \min\{\varepsilon,\, 1 - \|\bar{v}\|\}.$$

Observe that with these choices, if $t \in [0,\, t_0]$, $w^1 \in B(0,\, \theta)$, and if $(v^1,\, v^2) \in \mathcal{V}(\rho)$, where

$$\mathcal{V}(\rho) = (K_1 \cap B(v^0,\, \rho)) \times (K_2 \cap B(0,\, \rho))\},$$

then

$$u(t,\, w^1,\, v^1,\, v^2) \in K \cap \operatorname{cone} B(\bar{v},\, \varepsilon), \quad \|u(t,\, w^1,\, v^1,\, v^2)\| \leq 1. \tag{2.8}$$

Define the mapping $A : K_1 \to \operatorname{span} \Phi'(\bar{u})K = \operatorname{span} \Phi'(\bar{u})K_1$ (see (2.5)) by setting $A(v^1) = \Phi'(\bar{u})v^1$ for $v^1 \in K_1$. Since $\bar{w} \in \operatorname{ri} \Phi'(\bar{u})K_1$, employing [2, Corollary 2] we easily deduce that A covers at a linear rate with respect to $K_1 \cap B(v^0,\, \rho)$, provided $\rho > 0$ is small enough.

Furthermore, let Π be an orthogonal projector onto $(\operatorname{span} \Phi'(\bar{u})K)^{\perp}$. Define the mapping $B : K_2 \to (\operatorname{span} \Phi'(\bar{u})K)^{\perp}$ by setting $B(v^2) = \Pi \Phi''(\bar{u})[\bar{v},\, v^2]$ for $v^2 \in K_2$. Condition (2.6) implies that B is surjective, and according to Theorem 2.1, B covers at a linear rate.

Combining these two covering properties, it can be readily seen that the mapping $C : K_1 \times K_2 \to \mathbb{R}^q$, $C(v^1,\, v^2) = \Phi'(\bar{u})v^1 + \Phi''(\bar{u})[\bar{v},\, v^2]$ covers at a linear rate with some constant $\theta_2 > 0$ with respect to $\mathcal{V}(\rho)$.

After some estimations employing the mean-value theorem, we further obtain that

$$\begin{aligned} \Phi(\bar{u} + tu(t,\, w^1,\, v^1,\, v^2)) \;&=\; t\Phi'(\bar{u})(\bar{v} + v(w^1) + tv^1) + \frac{1}{2} t^2 \Phi'(\bar{u})[\bar{v},\, \bar{v}] + t^2 \Phi''(\bar{u})[\bar{v},\, v^2] \\ &\quad + \omega(t,\, w^1,\, (v^1,\, v^2)), \end{aligned} \tag{2.9}$$

where the mapping $\omega : \mathbb{R} \times \mathbb{R}^q \times (\mathbb{R}^p \times \mathbb{R}^p) \to \mathbb{R}^q$ satisfies the following properties, perhaps after further reducing $t_0 > 0$, $\theta > 0$, and $\rho > 0$: for all $t \in [0,\, t_0]$ and all $w^1 \in B(0,\, \theta)$

$$\|\omega(t,\, w^1,\, (v^0,\, 0))\| \leq \frac{1}{4} \theta_2 t^2 \rho, \tag{2.10}$$

and $\omega(t,\, w^1,\, \cdot)$ is Lipschitz-continuous on $\mathcal{V}(\rho)$ with a constant $\theta_2 t^2/2$.

By (2.9), equation (2.7) with u of the specified form can be written as

$$t^2(\Phi'(\bar{u})v^1 + \Phi''(\bar{u})[\bar{v},\, v^2]) + \omega(t,\, w^1,\, (v^1,\, v^2)) = t^2(\bar{w} + w^2),$$

or equivalently, for $t > 0$,

$$\Psi(v^1,\, v^2) = \Omega(t,\, w^1,\, w^2,\, (v^1,\, v^2)), \tag{2.11}$$

where we define $\Psi : K_1 \times K_2 \to \mathbb{R}^q$,

$$\Psi(v^1,\, v^2) = C(v^1,\, v^2) - \bar{w},$$

and $\Omega(t,\, w^1,\, w^2,\, \cdot) : K_1 \times K_2 \to \mathbb{R}^q$,

$$\Omega(t,\, w^1,\, w^2,\, (v^1,\, v^2)) = w^2 - \frac{1}{t^2}\omega(t,\, w^1,\, (v^1,\, v^2)).$$

Observe that $C(v^0,\, 0) = \bar{w}$, and further reducing $\rho > 0$ if necessary, and then also reducing $\theta > 0$ so that $\theta < \theta_2\rho/4$, we obtain from (2.10) that for all $t \in (0,\, t_0]$, and all $w^1 \in B(0,\, \theta) \cap C$ and $w^2 \in B(0,\, \theta)$, it holds that

$$\|\Omega(t,\, w^1,\, w^2,\, (v^0,\, 0))\| \leq \|w^2\| - \frac{1}{t^2}\|\omega(t,\, w^1,\, (v^0,\, 0))\| < \frac{1}{2}\theta_2\rho.$$

Since Ψ is covering with a constant θ_2 with respect to $\mathcal{V}(\rho)$, while $\Omega(t,\, w^1,\, w^2,\, \cdot)$ is Lipschitz-continuous on $\mathcal{V}(\rho)$ with a constant $\theta_2/2$, we obtain from Proposition 2.1 that equation (2.11) has a solution $(v^1,\, v^2) \in \mathcal{V}(\rho)$. Then $u = \bar{u} + tu(t,\, w^1,\, v^1,\, v^2)$ solves equation (2.7), and (2.8) holds, where the second relation then implies that $u \in B(\bar{u},\, t)$. This gives the needed conclusion.

The last assertion of the theorem evidently follows from the proof above, taking into account the last assertion of Theorem 2.1. ■

Theorem 2.2 is an improvement over [2, Theorem 2′], where a more restrictive condition was used instead of (2.1), namely

$$\operatorname{span} \Phi'(\bar{u})K + \Phi''(\bar{u})[\bar{v},\, \ker \Phi'(\bar{u}) \cap K] = \mathbb{R}^q. \tag{2.12}$$

As will be discussed below, condition (2.1) is significantly weaker, and Theorem 2.2 covers a much wider area than [2, Theorem 2′].

In the case when $K = \mathbb{R}^p$ (unconstrained equation), Theorem 2.2 is closely related to [22, Theorem 4], which, in its turn, follows from the results in [20].

Consider first the case when Robinson's condition holds, i.e., $\Phi'(\bar{u})K = \mathbb{R}^q$. Then the 2-regularity condition (2.1) (as well as (2.12)) holds automatically for all $\bar{v} \in \mathbb{R}^q$, including $\bar{v} = 0$, and one can take $C = \mathbb{R}^q$ and $\bar{w} = 0$. With these choices, Theorem 2.2 gives the classical covering result (which is a consequence of Robinson's stability theorem [30]): for every $w \in B(0,\, \theta t_0)$, there exists $u(w) \in \bar{u} + K$ such that $\Phi(u(w)) = w$, and $\|u(w) - \bar{u}\| \leq \|w\|/\theta$.

Observe, however, that if $p = q$, then Robinson's condition can only hold when $K = \mathbb{R}^p$, i.e., in the case of an unconstrained equation.

When Robinson's condition does not hold, one cannot expect to cover the entire neighborhood of 0. The union of the sets in the left-hand sides of (2.2) over all $t \in [0,\, t_0]$ is, in general, a "horn" with a "spike" at 0, and Theorem 2.2 says that under its assumptions, this "horn" is covered by Φ on $\bar{u} + K$: for every w in this "horn", there exists $u(w) \in \bar{u} + K$ such that $\Phi(u(w)) = w$, and $\|u(w) - \bar{u}\| = O(t_0)$ as $t_0 \to 0$. The next question is when this "horn" can be guaranteed to be "large", i.e., not "asymptotically thin", or in other words, when the cone of feasible directions to this set at 0 has a nonempty interior. We next demonstrate two such cases.

Lemma 2.1 *Let $C \subset \mathbb{R}^q$ be any closed cone, and $\tilde{C} \subset \mathbb{R}^q$ be any cone, such that $C \setminus \{0\} \subset \operatorname{int} \tilde{C}$.*

Then for any $\bar{w} \in \mathbb{R}^q$, any $\tilde{\theta} > 0$, and any $\theta \in (0,\, \tilde{\theta})$, there exists $t_0 > 0$ such that for every $t \in [0,\, t_0]$, it holds that

$$S(0,\, \theta t) \cap C \subset B(0,\, \tilde{\theta} t) \cap \tilde{C} + t^2 \bar{w}.$$

Proof. If $\bar{w} = 0$, this statement holds trivially. Let $\bar{w} \neq 0$.

By compactness of the set $S(0,\, \theta) \cap C$, and by the assumptions on C and $\tilde{C}$, there exists $\delta \in (0,\, \tilde{\theta} - \theta]$ such that $S(0,\, \theta) \cap C + B(0,\, \delta) \subset \tilde{C}$. For any $w \in S(0,\, \theta) \cap C$ and any $t \geq 0$, set $\tilde{w} = w - t\bar{w}$. Then, assuming that $t\|\bar{w}\| \leq \delta$, we obtain $\tilde{w} \in B(w,\, t\|\bar{w}\|) \subset B(w,\, \delta) \subset B(0,\, \theta + \delta) \cap \tilde{C} \subset B(0,\, \tilde{\theta}) \cap \tilde{C}$. It remains to observe that $tw = t\tilde{w} + t^2\bar{w} \in B(0,\, \tilde{\theta} t) \cap \tilde{C} + t^2 \bar{w}$, and hence, the needed assertion holds with $t_0 = \delta / \|\bar{w}\|$. ∎

Corollary 2.1 *Let Φ be twice differentiable near $\bar{u} \in \Phi^{-1}(0)$, with its second derivative being continuous at $\bar{u}$, and let*

$$\operatorname{int} \Phi'(\bar{u})K \neq \emptyset. \tag{2.13}$$

Then for any closed convex cone $C \subset \mathbb{R}^q$ satisfying $C \setminus \{0\} \subset \operatorname{int} \Phi'(\bar{u})K$, there exists $\theta > 0$ such that for every $w \in C$ close enough to 0, there exists $u(w) \in \bar{u} + K$ such that $\Phi(u(w)) = w$, and $\|u(w) - \bar{u}\| \leq \|w\|/\theta$.

Proof. By compactness of the set $S(0,\, \theta) \cap C$, and by the assumptions on C, there exists $\delta > 0$ such that $S(0,\, \theta) \cap C + B(0,\, \delta) \subset \operatorname{int} \Phi'(\bar{u})K$, and hence, $\tilde{C} = \operatorname{cone}(S(0,\, \theta) \cap C + B(0,\, \delta))$ is a closed convex cone satisfying $\tilde{C} \setminus \{0\} \subset \operatorname{int} \Phi'(\bar{u})K$ and $C \setminus \{0\} \subset \operatorname{int} \tilde{C}$.

Furthermore, condition (2.13) is equivalent to the equality $\operatorname{span} \Phi'(\bar{u})K = \mathbb{R}^q$, and therefore, the 2-regularity condition (2.12) again holds automatically for all $\bar{v} \in \mathbb{R}^q$, including $\bar{v} = 0$ (as well as in the case when Robinson's condition holds).

Fix any $\bar{w} \in \operatorname{int} \Phi'(\bar{u})K$. Since

$$\varphi_{0,\, \bar{w}}(t) + \Gamma_{\tilde{C},\, \theta}(t) = B(0,\, \theta t) \cap \tilde{C} + t^2 \bar{w} + B(0,\, \theta t^2),$$

applying Theorem 2.2 with C replaced by $\tilde{C}$, we obtain the existence of $\tilde{\theta} > 0$ and $\tilde{t}_0 > 0$ such that

$$B(0,\, \tilde{\theta}t) \cap \tilde{C} + t^2\bar{w} + B(0,\, \tilde{\theta}t^2) \subset \Phi((\bar{u} + K) \cap B(\bar{u},\, t))$$

for all $t \in [0,\, \tilde{t}_0]$. Then by Lemma 2.1, for any $\theta \in (0,\, \tilde{\theta})$, there exists $t_0 \in (0,\, \tilde{t}_0]$ such that

$$S(0,\, \theta t) \cap C \subset \Phi((\bar{u} + K) \cap B(\bar{u},\, t))$$

for all $t \in [0,\, t_0]$.

Now, for any $w \in C$, we have that $w \in S(0,\, \theta t) \cap C$ with $t = \|w\|/\theta$, and hence, $w \in \Phi((\bar{u} + K) \cap B(\bar{u},\, t))$ provided $\|w\| \leq \theta t_0$. This mean that for any such w, there exists $u(w) \in \bar{u} + K$ such that $\Phi(u(w)) = w$, and $\|u(w) - \bar{u}\| \leq t = \|w\|/\theta$, completing the proof. ∎

Remark 2.1 Essentially the same result was obtained directly (i.e., not as a corollary of Theorem 2.2) in [29, Theorem 3]. In particular, twice differentiability of Φ is in fact not needed in this result: it is enough to assume that Φ is strictly differentiable at $\bar{u}$.

Observe that in Corollary 2.1, C can always be chosen in such a way that $\operatorname{int} C \neq \emptyset$, and hence, in this case, the corresponding set of "good" right-hand side perturbations (those "survived" by $\bar{u}$) is "large", as, according to this corollary, the cone of feasible directions to this set at 0 contains C.

Condition (2.13) can only hold when $\operatorname{rank} \Phi'(\bar{u}) = q$, or, in other words, $\operatorname{im} \Phi'(\bar{u}) = \mathbb{R}^q$ (and is automatic if, in addition, $\operatorname{int} K \neq \emptyset$). In particular, unlike Corollary 2.2 below, Corollary 2.1 is never applicable in the case of unconstrained equations when the solution in question is singular.

Remark 2.2 Consider the case when $p = q$, which will be the case of interest in Section 3. In this setting Corollary 2.1 becomes trivial in a sense that it is an easy consequence of classical results. Indeed, (2.13) may only hold when $\Phi'(\bar{u})$ is nonsingular. Therefore, by the classical inverse function theorem, for all $w \in \mathbb{R}^p$ close to 0, equation $\Phi(u) = w$ has near $\bar{u}$ the unique solution $u(w)$, which depends smoothly on w, and $u'(0) = (\Phi'(\bar{u}))^{-1}$. This implies, in particular, the existence of $\theta > 0$ such that $\|u(w) - \bar{u}\| \leq \|w\|/\theta$. Furthermore, substituting $u = u(w)$ into the constraint $u \in \bar{u} + K$ gives the characterization of "good" perturbations w as those satisfying:

$$u(w) \in \bar{u} + K.$$

Taking into account the appearance of the Jacobian $u'(0)$, it follows that every $\omega \in \operatorname{int} \Phi'(\bar{u})K$ is a feasible direction to the set of "good" perturbations at 0, and the needed conclusion is now evident.

This reasoning also applies in the context of Theorem 2.1: if $p = q$ and A is nonsingular, the assertion of this theorem holds trivially for $C = AK$.

Corollary 2.2 *Let Φ be twice differentiable near $\bar{u} \in \Phi^{-1}(0)$, with its second derivative being continuous at $\bar{u}$, and let there exist $\bar{v} \in \ker \Phi'(\bar{u}) \cap K$ such that Φ is 2-regular at $\bar{u}$ with respect to K in the direction $\bar{v}$, i.e.,* (2.1) *holds.*

Then for every $\bar{w} \in \operatorname{ri} \Phi'(\bar{u})K$ there exists $\theta > 0$ such that for every

$$w \in \operatorname{cone} B\left(\frac{1}{2}\Phi''(\bar{u})[\bar{v},\, \bar{v}] + \bar{w},\, \theta\right) \tag{2.14}$$

close enough to 0*, there exists $u(w) \in \bar{u} + K$ such that $\Phi(u(w)) = w$, and $\|u(w) - \bar{u}\| \leq \|w\|^{1/2}/\theta$.*

Proof. Applying Theorem 2.2 with $C = \{0\}$, we obtain the existence of $\theta > 0$ and $t_0 > 0$ such that for all $t \in [0,\, t_0]$ it holds that

$$t^2\left(\frac{1}{2}\Phi''(\bar{u})[\bar{v},\, \bar{v}] + \bar{w}\right) + B(0,\, \theta t^2) \subset \Phi((\bar{u} + K) \cap B(\bar{u},\, t)).$$

The union of the sets in the left-hand side over all $t \geq 0$ coincides with the cone in the right-hand side of (2.14), and the needed property evidently follows. ■

In the case of an unconstrained equation, Corollary 2.2 is closely related to [22, Theorem 5]. Observe also that if $\operatorname{int} K \neq \emptyset$, then $\operatorname{span} \Phi'(\bar{u})K = \operatorname{im} \Phi'(\bar{u})$. Therefore, in the important special case when $\bar{v} \in \operatorname{int} K$ (which is automatic for an unconstrained equation), due to the equality $R_K(\bar{v}) = \mathbb{R}^p$, condition (2.1) takes the form (1.5), which is the 2-regularity condition as defined for the unconstrained case (or in other words, with respect to the entire $\mathbb{R}^p$). In particular, unlike it would have been for (2.12), K does not appear in this condition anymore. Observe further that if $\bar{v} \in \operatorname{int} K$, the result of Theorem 2.2 can be derived from [16, Remark 5].

The next example demonstrates that if $\bar{v} \notin \operatorname{int} K$, then (2.1) cannot be replaced by the weaker (1.5).

Example 2.2 Consider $\Phi : \mathbb{R}^3 \to \mathbb{R}^2$, $\Phi(u) = (u_1u_2 + u_1^4 + u_2^4,\, u_2u_3)$, $K = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$, $\bar{u} = 0$.

We have: $\Phi'(\bar{u}) = 0$, and for $\bar{v} = (0,\, 1,\, \nu)$ with $\nu \in \mathbb{R}$ it holds that $R_K(\bar{v}) = \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$ and

$$\Phi''(\bar{u})[\bar{v}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \nu & 1 \end{pmatrix},$$

implying that $\Phi''(\bar{u})[\bar{v},\, R_K(\bar{v})] = \mathbb{R}_+ \times \mathbb{R}$, and hence, (2.1) does not hold, unlike (1.5).

Observe further that $\varphi_{\bar{v},\,0}(t) = t^2\Phi''(\bar{u})[\bar{v},\, \bar{v}]/2 = (0,\, t^2\nu)$, and if $\nu \neq 0$, it can be easily seen that the equation

$$\Phi(u) = \varphi_{\bar{v},\,0}(t)$$

does not have solutions in K for any $t \neq 0$. The reason is that $\bar{v} \notin \operatorname{int} K$.

We now again turn our attention to the case when $p = q$.

Remark 2.3 If $p = q$, (2.1) may hold for some $\bar{v}$ only when

$$\operatorname{span} \Phi'(\bar{u}) R_K(\bar{v}) = \operatorname{im} \Phi'(\bar{u}), \quad \ker \Phi'(\bar{u}) \subset R_K(\bar{v}), \tag{2.15}$$

for the similar reason why Robinson's condition can only hold when $K = \mathbb{R}^p$. Therefore, in this case, the constrained 2-regularity condition (2.1) is equivalent to the combination of (2.15) and the the unconstrained 2-regularity condition (1.5).

As discussed above, the equality in (2.15) is automatic if $\operatorname{int} K \neq \emptyset$. The inclusion in (2.15) implies that $\ker \Phi'(\bar{u}) \cap R_K(\bar{v}) = \ker \Phi'(\bar{u})$. The need of (2.15) limits applicability of Corollary 2.2 when $p = q$, but much less than it would have been if using condition (2.12) instead of (2.1).

Furthermore, if

$$\ker \Phi'(\bar{u}) \cap K \subset T_{\Phi^{-1}(0)}(\bar{u}), \tag{2.16}$$

which is automatic provided the constrained local Lipschitzian error bound (1.8) holds at $\bar{u}$, then for every $\bar{v} \in \ker \Phi'(\bar{u}) \cap K$ it holds that $\Phi''(\bar{u})[\bar{v}, \bar{v}] \in \operatorname{im} \Phi'(\bar{u})$, implying that Φ cannot be 2-regular (and even less so 2-regular with respect to K) at $\bar{u}$ in such direction $\bar{v} \neq 0$ if $p = q$. Therefore, similarly to the unconstrained case [22], Corollary 2.2 is not applicable at a singular solution when $p = q$ and (2.16) holds. Observe that (2.16) is automatic if $\bar{u}$ is a noncritical solution of the unconstrained equation (1.3).

3 Complementarity problems

We now turn our attention to a nonlinear complementarity problem (NCP)

$$z \geq 0, \quad F(z) \geq 0, \quad \langle z, F(z) \rangle = 0, \tag{3.1}$$

with a sufficiently smooth mapping $F : \mathbb{R}^s \to \mathbb{R}^s$. If F is affine, i.e., $F(z) = Mz + c$ with some $M \in \mathbb{R}^{s \times s}$ and $c \in \mathbb{R}^s$, (3.1) is a linear complementarity problem (LCP). Along with (3.1), we will consider its perturbed version

$$z \geq 0, \quad F(z) - y \geq 0, \quad \langle z, F(z) - y \rangle = 0, \tag{3.2}$$

where $y \in \mathbb{R}^s$ characterizes perturbations of F.

For a given solution $\bar{z}$ of (3.1), define the index sets

$$\begin{aligned} I_0 = I_0(\bar{z}) &= \{i = 1, \ldots, s \mid \bar{z}_i = F_i(\bar{z}) = 0\}, \\ I_1 = I_1(\bar{z}) &= \{i = 1, \ldots, s \mid \bar{z}_i > 0, \ F_i(\bar{z}) = 0\}, \\ I_2 = I_2(\bar{z}) &= \{i = 1, \ldots, s \mid \bar{z}_i = 0, \ F_i(\bar{z}) > 0\}. \end{aligned}$$

Then, near $\bar{z}$, and for y close enough to 0, the solutions set of (3.2) is the union of solution sets of the branch systems

$$z_{J_1} \geq 0, \quad F_{J_1}(z) = y_{J_1}, \quad z_{J_2} = 0, \quad F_{J_2}(z) \geq y_{J_2}, \quad z_{I_2} = 0, \quad F_{I_1}(z) = y_{I_1}, \tag{3.3}$$

defined by all partitions (J_1, J_2) of I_0, i.e., pairs of index sets satisfying $J_1 \cup J_2 = I_0$, $J_1 \cap J_2 = \emptyset$.

Our analysis below strongly relies on this piecewise decomposition of the solution set. That said, we emphasize that there exist different approaches to stability analysis for complementarity and more general variational problems, e.g., those treating the solution set as a whole by means of contemporary variational analysis. Among the most prominent tools of the latter is the Mordukhovich criterion for metric regularity of multifunctions [28, Theorem 4.18, Remark 4.78], allowing for exact verifiable characterizations of this important stability property. Recall, however, in this paper we focus on those cases when metric regularity (which for NCP is the same as strong metric regularity; see [10]) does not hold, and when one cannot expect stability with respect to all small perturbations. Specifically, we are mostly interested in the cases of nonisolated solutions, which can never be (strongly) metrically regular, and our goal is to single out those special solutions which can still be guaranteed to be stable with respect to "large" sets of perturbation, while the other solutions are normally "killed" by generic perturbations. In particular, we obtain weaker than metric regularity stability results (not for all small perturbations, and with square-root estimates, in general) but under much weaker assumptions. We believe that the examples presented below in this section cannot be tackled by any simpler tools known so far.

If the solution $\bar{z}$ satisfies the strict complementarity condition $I_0(\bar{z}) = \emptyset$, (3.3) reduces to the system of equations

$$z_{I_2} = 0, \quad F_{I_1}(z) = y_{I_1}.$$

Therefore, assuming that the components of z are ordered in such a way that $z = (z_{I_1},\, z_{I_2})$, the unknown component z_{I_1} must satisfy

$$F_{I_1}(z_{I_1},\, 0) = y_{I_1}, \tag{3.4}$$

and the behavior of such solution $\bar{u}$ is characterized by the results from [22], discussed in Section 1, applied to this unconstrained equation.

The case when strict complementarity is violated is a whole different story. Let $\mathcal{S}$ stand for the solution set of (3.1). According to [13, Lemma 1, Theorem 2], and similarly to the case of an unconstrained equation (1.3), the local Lipschitzian error bound of the form

$$\operatorname{dist}(z,\, \mathcal{S}) = O(\|\min\{z,\, F(z)\}\|) \tag{3.5}$$

as $z \in \mathbb{R}^s$ tends to $\bar{z}$, with min applied componentwise, is equivalent to the upper Lipschitzian property, which consists of saying that

$$\operatorname{dist}(z(y),\, \mathcal{S}) = O(\|y\|)$$

as $y \in \mathbb{R}^s$ tends to 0, where $z(y)$ is any solution of the perturbed NCP (3.2) close enough to $\bar{z}$. However, the next simple example demonstrates that unlike in the case of unconstrained equation (1.3), these equivalent properties do not necessarily imply the lack of stability of the solution in question if it does not satisfy strict complementarity, even when this solution is nonisolated.

Example 3.1 Consider the NCP (3.1) with $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F(z) = (z_2,\, z_1)$. The solution set has the form $\mathcal{S} = \{z \in \mathbb{R}^2 \mid z_1 \geq 0,\, z_2 \geq 0,\, z_1 z_2 = 0\}$. Since this NCP is actually an

LCP, it follows from [32] that the upper Lipschitzian property, and hence, the error bound (3.5) is satisfied at every solution of this problem, including $\bar{z} = 0$, which is the only solution violating strict complementarity.

The perturbed NCP (3.2) has the following four groups of solutions, corresponding to four branch systems (3.3) at $\bar{z}$:

1. For $J_1 = \emptyset$, $J_2 = \{1, 2\}$, we have the solution $z = 0$ when $y_1 \leq 0$, $y_2 \leq 0$.
2. For $J_1 = \{1\}$, $J_2 = \{2\}$, solutions exist when $y_1 = 0$, and they are those z satisfying $z_1 \geq \max\{0, y_2\}$, $z_2 = 0$.
3. For $J_1 = \{2\}$, $J_2 = \{1\}$, solutions exist when $y_2 = 0$, and they are those z satisfying $z_1 = 0$, $z_2 \geq \max\{0, y_1\}$.
4. For $J_1 = \{1, 2\}$, $J_2 = \emptyset$, we have the solution $z_1 = y_2$, $z_2 = y_1$ when $y_1 \geq 0$, $y_2 \geq 0$.

Solutions of groups 2 and 3 may approximate as $y \to 0$ any point in $\mathcal{S}$ satisfying strict complementary, but these solutions exist for very special perturbations only, which agrees with the discussion above: those strictly complementary solutions are noncritical in the sense of [22] applied to the corresponding unconstrained equation (3.4), or in other words, the error bound (1.6) holds at them.

At the same time, solutions of groups 1 and 4 tend to $\bar{z}$ as $y \to 0$, and they exist for wide ranges of y.

Therefore, *solutions violating strict complementarity may have special stability properties regardless of the presence or absence of the local Lipschitzian error bound at them, and in this sense, they can be all regarded as critical.* This point of view is further supported by the following observations. Critical solutions of unconstrained equations are known to be specially attractive for iterative sequences generated by Newton-type methods [23]. That said, [24, Example 7.9] demonstrates the case when sequences of the sequential quadratic programming algorithm are attracted by a prima-dual solution of the Karush–Kuhn–Tucker (KKT) system for an optimization problem with inequality constraints, with nonunique dual part violating strict complementary, but noncritical in the sense of [24, Definition 1.41], which is the same as saying that the local Lipschitzian error bound and the equivalent upper Lipschitzian property hold at it [24, Proposition 1.43].

However, of course, violation of strict complementary by itself does not imply stability, even in case of LCP.

Example 3.2 Consider the LCP (3.1) with $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F(z) = (0, -z_1 + z_2 + 1)$. The solution set has the form $\mathcal{S} = \{z \in \mathbb{R}^2 \mid 0 \leq z_1 \leq 1,\ z_2 = 0\} \cup \{z \in \mathbb{R}^2 \mid z_1 = z_2 + 1,\ z_2 \geq 0\}$.

A solution $\bar{z} = (1, 0)$ is nonisolated, and violates strict complementarity unlike all other solutions close to it. Evidently, $\bar{z}$ can be stable only subject to perturbations $y \in \mathbb{R}^2$ with $y_1 = 0$.

In the rest of this section, we obtain some insight into special stability properties of a solution $\bar{z}$ of (3.1), violating strict complementarity. For a given partition (J_1, J_2) of

$I_0 = I_0(\bar{z})$, let the components of z be ordered in such a way that $z = (z_{I_1},\, z_{I_2},\, z_{J_1},\, z_{J_2})$. Introducing the slack variable $\sigma \in \mathbb{R}^{|J_2|}$, pairs $(z_{I_1},\, z_{J_1})$ satisfying (3.3) are equivalently characterized by the system

$$\begin{gathered} F_{I_1}(z_{I_1},\, 0,\, z_{J_1},\, 0) = y_{I_1}, \quad F_{J_1}(z_{I_1},\, 0,\, z_{J_1},\, 0) = y_{I_1}, \quad F_{J_2}(z_{I_1},\, 0,\, z_{J_1},\, 0) - \sigma = y_{J_2}, \\ z_{J_1} \geq 0, \quad \sigma \geq 0, \end{gathered} \tag{3.6}$$

with respect to $u = (z_{I_1},\, z_{J_1},\, \sigma)$. System (3.6) is a constrained equation

$$\Phi(u) = w, \quad u \in K, \tag{3.7}$$

where $\Phi : \mathbb{R}^{|I_1|} \times \mathbb{R}^{|J_1|} \times \mathbb{R}^{|J_2|} \to \mathbb{R}^{|I_1|} \times \mathbb{R}^{|J_1|} \times \mathbb{R}^{|J_2|}$,

$$\Phi(u) = (F_{I_1}(z_{I_1},\, 0,\, z_{J_1},\, 0),\, F_{J_1}(z_{I_1},\, 0,\, z_{J_1},\, 0),\, F_{J_2}(z_{I_1},\, 0,\, z_{J_1},\, 0) - \sigma), \tag{3.8}$$

$$K = \mathbb{R}^{|I_1|} \times \mathbb{R}_+^{|J_1|} \times \mathbb{R}_+^{|J_2|}, \tag{3.9}$$

with the right-hand side perturbation $w = (y_{I_1},\, y_{J_1},\, y_{J_2})$. The basic solution of interest (of (3.7) with $w = 0$) is $\bar{u} = (\bar{z}_{I_1},\, 0,\, 0)$, and (3.7) is equivalent to

$$\Phi(u) = w, \quad u \in \bar{u} + K. \tag{3.10}$$

3.1 Linear complementarity problem

Let $F(z) = Mz + c$ with some $M \in \mathbb{R}^{s \times s}$ and $c \in \mathbb{R}^s$, i.e., let (3.1) be an LCP. We first consider how Theorem 2.1 can be applied in this context. This leads to a result different from stability and sensitivity results existing in the LCP literature. The latter usually deals with all perturbations which are only supposed to be small enough, but either for the case of an isolated solution of the unperturbed problem, or they are concerned with the behavior of the entire solution set [8, Chapter 7]. Here, we are interested in stability properties of a particular solution which can be nonisolated, but nevertheless, can be guaranteed to "survive" large classes of perturbations, even though those classes cannot be expected to contain a full neighborhood of 0.

For LCP, the mapping Φ defined in (3.8) is also affine: $\Phi(u) = Au + b$, where

$$A = \begin{pmatrix} M_{I_1 I_1} & M_{I_1 J_1} & 0 \\ M_{J_1 I_1} & M_{J_1 J_1} & 0 \\ M_{J_2 I_1} & M_{J_2 J_1} & -I \end{pmatrix}, \quad b = \begin{pmatrix} c_{I_1} \\ c_{J_1} \\ c_{J_2} \end{pmatrix}.$$

Set

$$M_1 = M_1(\bar{z};\, J_1,\, J_2) = \begin{pmatrix} M_{I_1 I_1} & M_{I_1 J_1} \\ M_{J_1 I_1} & M_{J_1 J_1} \end{pmatrix}, \tag{3.11}$$

$$M_2 = M_2(\bar{z};\, J_1,\, J_2) = \begin{pmatrix} M_{J_2 I_1} & M_{J_2 J_1} \end{pmatrix}. \tag{3.12}$$

Since K is polyhedral, Theorem 2.1 ensures stability of $\bar{u}$ with respect to right-hand side perturbations $w \in AK$, and with Lipschitzian estimate. The set AK is a cone, and it is "large" in our sense if $\operatorname{int} AK \neq \emptyset$. Since $\operatorname{int} K \neq \emptyset$, the last condition is equivalent to nonsingularity of A, which, in its turn, is equivalent to nonsingularity of the matrix M_1. Therefore, if this matrix is nonsingular for at least one partition $(J_1,\ J_2)$ of I_0 (with a convention that an empty matrix is nonsingular), solution $\bar{z}$ is stable subject to a "large" set AK of perturbations: this set is a polyhedral cone with a nonempty interior, which can be explicitly characterized by the inequalities

$$\left(M_1^{-1}\left(\begin{array}{c} y_{I_1} \\ y_{J_1} \end{array}\right)\right)_{J_1} \geq 0, \quad M_2 M_1^{-1}\left(\begin{array}{c} y_{I_1} \\ y_{J_1} \end{array}\right) - y_{J_2} \geq 0. \tag{3.13}$$

Proposition 3.1 *Let $\bar{z}$ be a solution of the LCP* (3.1) *with $F(z) = Mz + c$, $M \in \mathbb{R}^{s \times s}$, and $c \in \mathbb{R}^s$. Let there exist a partition $(J_1,\ J_2)$ of $I_0 = I_0(\bar{z})$ such that the matrix M_1 defined in* (3.11) *is nonsingular.*

Then the polyhedral cone of $y \in \mathbb{R}^s$ satisfying (3.13) *with M_2 defined in* (3.12) *has a nonempty interior, and there exists $\theta > 0$ such that for every y in this cone, there exists a solution $z(y)$ of* (3.2) *satisfying $\|z(y) - \bar{z}\| \leq \|y\|/\theta$.*

Observe that taking $J_1 = \emptyset$, $J_2 = I_0$, reduces M_1 to $M_{I_1,\,I_1}$, and in particular, if $I_1 = \emptyset$, then Proposition 3.1 is automatically applicable with the specified J_1 and J_2, and with (3.13) transforming into the inequality $y_{I_0} \leq 0$.

We have derived this proposition as an example of application of Theorem 2.1. However, in fact, Proposition 3.1 can be easily derived directly from (3.3), and this agrees with Remark 2.2.

For the LCP in Example 3.1 we have:

$$M = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right),$$

and $I_0(\bar{z}) = \{1,\ 2\}$, $I_1(\bar{z}) = I_2(\bar{z}) = \emptyset$. Consider again the partitions of I_0:

1. For $J_1 = \emptyset$, the matrix M_1 is empty, and hence, Proposition 3.1 is applicable, with (3.13) taking the form $y \leq 0$.
2. For $J_1 = \{1\}$, $J_2 = \{2\}$, it holds that $M_1 = 0$, and Proposition 3.1 is not applicable.
3. The case of $J_1 = \{2\}$, $J_2 = \{1\}$ is considered similarly to item 2, with the same conclusion.
4. For $J_1 = \{1,\ 2\}$, $J_2 = \emptyset$, it holds that $M_1 = M$, which is nonsingular, and hence, Proposition 3.1 is applicable, with (3.13) taking the form $y \geq 0$.

Furthermore, for LCP in Example 3.2 we have:

$$M = \left(\begin{array}{cc} 0 & 0 \\ -1 & 1 \end{array}\right),$$

and $I_0(\bar{z}) = \{2\}$, $I_1(\bar{z}) = \{1\}$, $I_2(\bar{z}) = \emptyset$. For $J_1 = \emptyset$, it holds that $M_1 = 0$, and hence, Proposition 3.1 is not applicable. For $J_1 = \{2\}$, this matrix equals M, which is a singular matrix, and hence, Proposition 3.1 is again not applicable. This agrees with the observation that $\bar{z}$ can be stable subject to very special perturbations only.

3.2 Nonlinear complementarity problem

Observe first that if $\bar{z}$ is a solution of (3.1), violating strict complementarity, then for every branch system (3.7), the corresponding K is a proper subset of $\mathbb{R}^{|I_1|} \times \mathbb{R}^{|J_1|} \times \mathbb{R}^{|J_2|}$, and hence, Robinson's regularity condition can never hold, even if $\bar{z}$ is an isolated solution. At the same time, at nonisolated solutions satisfying strict complementarity, it also cannot hold, for obvious reasons.

In the nonlinear case, in order to apply Corollary 2.1, we need to replace A in considerations above by the Jacobian

$$\Phi'(\bar{u}) = \begin{pmatrix} M_1 & 0 \\ M_2 & -I \end{pmatrix}, \tag{3.14}$$

where now

$$M_1 = M_1(\bar{z};\, J_1,\, J_2) = \begin{pmatrix} \dfrac{\partial F_{I_1}}{\partial z_{I_1}}(\bar{z}) & \dfrac{\partial F_{I_1}}{\partial z_{J_1}}(\bar{z}) \\ \dfrac{\partial F_{J_1}}{\partial z_{I_1}}(\bar{z}) & \dfrac{\partial F_{J_1}}{\partial z_{J_1}}(\bar{z}) \end{pmatrix}, \tag{3.15}$$

$$M_2 = M_2(\bar{z};\, J_1,\, J_2) = \begin{pmatrix} \dfrac{\partial F_{J_2}}{\partial z_{I_1}}(\bar{z}) & \dfrac{\partial F_{J_2}}{\partial z_{J_1}}(\bar{z}) \end{pmatrix}. \tag{3.16}$$

Again, since $\operatorname{int} K \neq \emptyset$, condition (2.13) is equivalent to nonsingularity of this Jacobian, which, in its turn, is equivalent to nonsingularity of M_1. Therefore, if this matrix is nonsingular for at least one partition $(J_1,\, J_2)$ of I_0, Corollary 2.1 guarantees that the solution $\bar{z}$ of the NCP (3.1) is stable subject to a "large" set of perturbations (see also Remark 2.1).

Proposition 3.2 *Let F be strictly differentiable at a solution $\bar{z}$ of the NCP* (3.1)*. Let there exist a partition $(J_1,\, J_2)$ of $I_0 = I_0(\bar{z})$ such that the matrix M_1 defined in* (3.15) *is nonsingular.*

Then the set of $y \in \mathbb{R}^s$ satisfying

$$\left(M_1^{-1}\begin{pmatrix} y_{I_1} \\ y_{J_1} \end{pmatrix}\right)_{J_1} > 0, \quad M_2 M_1^{-1}\begin{pmatrix} y_{I_1} \\ y_{J_1} \end{pmatrix} - y_{J_2} > 0 \tag{3.17}$$

with M_2 defined in (3.16) *is nonempty, and for any closed convex cone $C \subset \mathbb{R}^s$ such that $C \setminus \{0\}$ is contained in the set given by* (3.17)*, there exists $\theta > 0$ such that for every $y \in C$ close enough to* 0*, there exists a solution $z(y)$ of* (3.2) *satisfying $\|z(y) - \bar{z}\| \leq \|y\|/\theta$.*

Taking $J_1 = \emptyset$, $J_2 = I_0$, reduces the matrix M_1 to $\frac{\partial \Phi_{I_1}}{\partial u_{I_1}}(\bar{z})$, and in particular, if $I_1 = \emptyset$, then Proposition 3.2 is automatically applicable with the specified J_1 and J_2, and with (3.17) transforming into the inequality $y_{I_0} < 0$.

Now suppose that the matrix M_1 (and hence, $\Phi'(\bar{u})$) is singular, and therefore, Corollary 2.1 is not applicable. Then one needs to employ second derivatives and apply Corollary 2.2, when applicable.

By Remark 2.3, in order to verify the assumptions of Corollary 2.2 in this context we need to show that there exists $\bar{v} \in \ker \Phi'(\bar{u}) \cap K$ satisfying (1.5) and (2.15). The equality in (2.15) is automatic because $\operatorname{int} K \neq \emptyset$ (see (3.9)). From (3.14) it evidently follows that

$$\ker \Phi'(\bar{u}) = \{v = (\zeta^1,\, M_2\zeta^1) \mid \zeta^1 \in \ker M_1\}, \quad \operatorname{im} \Phi'(\bar{u}) = \operatorname{im} M_1 \times \mathbb{R}^{|J_2|}. \tag{3.18}$$

In particular, taking into account (3.9), the inclusion in (2.15) has the form

$$(\zeta^1,\, M_2\zeta^1) \in R_K(\bar{v}) \quad \forall\, \zeta^1 \in \ker M_1, \tag{3.19}$$

and $\ker \Phi'(\bar{u}) \cap K$ consists of $\bar{v} = (\bar{\zeta}^1,\, M_2\bar{\zeta}^1)$ with $\bar{\zeta}^1 = (\bar{\zeta}_{I_1},\, \bar{\zeta}_{J_1}) \in \mathbb{R}^{|I_1|} \times \mathbb{R}^{|J_1|}$ satisfying

$$\bar{\zeta}_{J_1} \geq 0, \quad M_1\bar{\zeta}^1 = 0, \quad M_2\bar{\zeta}^1 \geq 0. \tag{3.20}$$

Furthermore, let π be the orthogonal projector in $\mathbb{R}^{|I_1|} \times \mathbb{R}^{|J_1|}$ onto $(\mathrm{im}\, M_1)^{\perp}$, and define the linear operator $\Lambda(\bar{\zeta}^1) : \ker M_1 \to (\mathrm{im}\, M_1)^{\perp}$,

$$\Lambda(\bar{\zeta}^1) = \pi \left(\begin{array}{cc} \dfrac{\partial^2 F_{I_1}}{\partial z_{I_1}^2}(\bar{z})[\bar{\zeta}_{I_1}] + \dfrac{\partial^2 F_{I_1}}{\partial z_{I_1}\partial z_{J_1}}(\bar{z})[\bar{\zeta}_{J_1}] & \dfrac{\partial^2 F_{I_1}}{\partial z_{I_1}\partial z_{J_1}}(\bar{z})[\bar{\zeta}_{I_1}] + \dfrac{\partial^2 F_{I_1}}{\partial z_{J_1}^2}(\bar{z})[\bar{\zeta}_{J_1}] \\ \dfrac{\partial^2 F_{J_1}}{\partial z_{I_1}^2}(\bar{z})[\bar{\zeta}_{I_1}] + \dfrac{\partial^2 F_{J_1}}{\partial z_{I_1}\partial z_{J_1}}(\bar{z})[\bar{\zeta}_{J_1}] & \dfrac{\partial^2 F_{J_1}}{\partial z_{I_1}\partial z_{J_1}}(\bar{z})[\bar{\zeta}_{I_1}] + \dfrac{\partial^2 F_{J_1}}{\partial z_{J_1}^2}(\bar{z})[\bar{\zeta}_{J_1}] \end{array} \right). \tag{3.21}$$

It can be easily derived from (3.18) that (1.5) is equivalent to saying that $\Lambda(\bar{\zeta}^1)$ is nonsingular. Therefore, Corollary 2.2 is applicable if there exists ζ^1 satisfying (3.19) (with $\bar{v} = (\bar{\zeta}^1,\, M_2\bar{\zeta}^1)$), (3.20), and such that the linear operator $\Lambda(\bar{\zeta}^1)$ is nonsingular. This yields the following

Proposition 3.3 *Let F be twice differentiable near a solution $\bar{z}$ of the NCP* (3.1)*, with its second derivative being continuous at $\bar{z}$. Let there exist a partition $(J_1,\, J_2)$ of $I_0 = I_0(\bar{z})$ and $\bar{\zeta}^1 = (\bar{\zeta}_{I_1},\, \bar{\zeta}_{J_1}) \in \mathbb{R}^{|I_1|} \times \mathbb{R}^{|J_1|}$ satisfying* (3.20) *and such that* (3.19) *holds with $\bar{v} = (\bar{\zeta}^1,\, M_2\bar{\zeta}^1)$, and the linear operator $\Lambda(\bar{\zeta}^1) : \ker M_1 \to (\mathrm{im}\, M_1)^{\perp}$ defined by* (3.21) *is nonsingular.*

Then there exist a cone $C \subset \mathbb{R}^s$ with nonempty interior and $\theta > 0$ such that for every $y \in C$ close enough to 0, there exists a solution $z(y)$ of (3.2) *satisfying $\|z(y) - \bar{z}\| \leq \|y\|^{1/2}/\theta$.*

Example 3.3 Consider the NCP (3.1) with $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F(z) = ((z_1 - 1)z_2,\, (z_1 - 1)^2)$. The solution set has the form $\mathcal{S} = \{z \in \mathbb{R}^2 \mid z_1 = 1,\, z_2 \geq 0\} \cap \{z \in \mathbb{R}^2 \mid z_1 \geq 0,\, z_2 = 0\}$, and the two solutions violating strict complementarity are $(0,\, 0)$ and $(1,\, 0)$. For the former, $I_1 = \emptyset$, and hence, Corollary 2.1 is applicable, giving stability subject to a "large" set of perturbations.

Consider $\bar{z} = (1,\, 0)$. Then $I_0(\bar{z}) = \{2\}$, $I_1(\bar{z}) = \{1\}$, $I_2(\bar{z}) = \emptyset$. Consider the branch systems (3.10) (in both of them $\bar{u} = (1,\, 0)$, $K = \mathbb{R} \times \mathbb{R}_+$, $w = y$):

1. For $J_1 = \emptyset$, $J_2 = \{2\}$, we have $F_{I_1}(z) = 0$ when $z_{J_2} = z_2 = 0$, implying that the matrix in (3.21) is always singular, and hence, 2-regularity cannot hold. The branch system (3.3) may have solutions tending to $\bar{z}$ only for perturbations $y \in \mathbb{R}^2$ with $y_1 = 0$.

2. For $J_1 = \{2\}$, $J_2 = \emptyset$ we have $M_1 = 0$, M_2 is empty, and hence, (3.20) holds for any $\bar{\zeta}^1 = \bar{\zeta} \in \mathbb{R}^2$ with $\bar{\zeta}_2 \geq 0$. Furthermore,

$$\Lambda(\bar{\zeta}^1) = F''(\bar{z})[\bar{\zeta}] = \begin{pmatrix} \bar{\zeta}_2 & \bar{\zeta}_1 \\ 2\bar{\zeta}_1 & 0 \end{pmatrix},$$

and this matrix is nonsingular for any $\bar{\zeta}_1 \neq 0$. Moreover, any direction $\bar{v} = \bar{\zeta}$ with with $\bar{\zeta}_2 > 0$ belongs to $\mathrm{int}\, K$, implying (3.19), and hence, Proposition 3.3 is applicable.

The branch system (3.3) has solutions for all perturbations $y \in \mathbb{R}^2$ with $y_2 \geq 0$, except for those with $y_1 \neq 0$, $y_2 = 0$; these solutions are

$$z(y) = \begin{cases} (1 + \sqrt{y_2},\, y_1/\sqrt{y_2}) & \text{if } y_1 \geq 0, \\ (1 - \sqrt{y_2},\, -y_1/\sqrt{y_2}) & \text{otherwise,} \end{cases}$$

and they tend to $\bar{z}$ as $y \to 0$ provided $y_1 = o(\sqrt{y_2})$. Therefore, this branch indeed gives rise to a "large" set of perturbations "survived" by $\bar{z}$.

We next modify the example above in order to demonstrate the case when Proposition 3.3 is applicable with different partitions, with $\bar{v} \notin \operatorname{int} K$ and $\bar{v} \in \operatorname{int} K$, respectively.

Example 3.4 Consider the NCP (3.1) with $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F(z) = ((z_1-1)^2+(z_1-1)z_2,\, (z_1-1)^2)$. The solution set has the form $\mathcal{S} = \{z \in \mathbb{R}^2 \mid z_1 = 1,\, z_2 \geq 0\}$, and the only solution violating strict complementarity is $\bar{z} = (1,\, 0)$.

We have: $I_0(\bar{z}) = \{2\}$, $I_1(\bar{z}) = \{1\}$, $I_2(\bar{z}) = \emptyset$. Consider the branch systems (3.10) (in both of them $\bar{u} = (1,\, 0)$, $K = \mathbb{R} \times \mathbb{R}_+$, $w = y$):

1. For $J_1 = \emptyset$, $J_2 = \{2\}$, we have $F_{I_1}(z) = (z_1 - 1)^2$ when $z_{J_2} = z_2 = 0$, $M_1 = 0$, $M_2 = 0$, and hence, by (3.18), $\ker \Phi'(\bar{u}) = \mathbb{R} \times \{0\}$. Observe that $\ker \Phi'(\bar{u}) \cap \operatorname{int} K = \emptyset$, and hence, there is no appropriate $\bar{v} \in \operatorname{int} K$.

   Nevertheless, for any $\bar{\zeta}^1 = \bar{\zeta}_1 \in \mathbb{R}$ and for $\bar{v} = (\bar{\zeta}_1,\, 0)$ it holds that $R_K(\bar{v}) = \mathbb{R} \times \mathbb{R}_+$, and hence, (3.19) is satisfied. Moreover, (3.20) holds trivially, while

   $$\Lambda(\bar{\zeta}^1) = \frac{\partial^2 F_1}{\partial z_1^2}(\bar{z})\bar{\zeta}_1 = 2\bar{\zeta}_1$$

   is nonsingular (distinct from zero) provided $\bar{\zeta}_1 \neq 0$. Therefore, the solution $\bar{u}$ of this branch system (and hence, the solution $\bar{z}$ of the NCP) is stable subject to a "large" set of perturbations.

   And this is indeed the case, as the branch system (3.3) has solutions for all perturbations $y \in \mathbb{R}^2$ with $y_1 \geq \max\{0,\, y_2\}$, and these solutions are $z(y) = (1 \pm \sqrt{y_1},\, 0)$, tending to $\bar{z}$ as $y \to 0$. Observe that the specified set of "good" y agrees with what appears in the right-hand side of (2.14) in Corollary 2.2. Moreover, this example demonstrates, in particular, that one cannot take in that corollary $\bar{w} = 0$.

2. For $J_1 = \{2\}$, $J_2 = \emptyset$ we have $u = z$, $\Phi(u) = F(z)$, $\Phi'(\bar{u}) = 0$, $\ker \Phi'(\bar{u}) \cap K = \mathbb{R} \times \mathbb{R}_+ \neq \ker \Phi'(\bar{u})$, implying that Φ cannot be 2-regular at $\bar{u}$ with respect to K in the sense of (2.12) in any direction at all. Nevertheless, Proposition 3.3 is applicable, similarly to the corresponding part of Example 3.4.

   And indeed, this branch system (3.3) has solutions for all perturbations $y \in \mathbb{R}^2$ with $y_2 \geq 0$, except for those with $y_1 \neq 0$, $y_2 = 0$; these solutions are

   $$z(y) = \begin{cases} (1 + \sqrt{y_2},\, (y_1 - y_2)/\sqrt{y_2}) & \text{if } y_1 \geq y_2, \\ (1 - \sqrt{y_2},\, -(y_1 - y_2)/\sqrt{y_2}) & \text{otherwise,} \end{cases}$$

   and they tend to $\bar{z}$ as $y \to 0$ provided $y_1 = o(\sqrt{y_2})$. Therefore, this branch also gives rise to a large set of "good" perturbations.

We complete this section by observing that the discussion in it can be extended to mixed complementarity problems (at the price of making the exposition more cumbersome), and in particular, to KKT systems.

4 Generalized Nash equilibrium problem

In order to avoid too heavy notation, in this section we restrict ourselves to a generalized Nash equilibrium problem (GNEP) with two players and shared constraints only:

$$\begin{array}{lll} \text{minimize}_{x^1} & f_1(x^1,\, x^2) & \qquad \text{minimize}_{x^2} & f_2(x^1,\, x^2) \\ \text{subject to} & g(x^1,\, x^2) \leq 0, & \qquad \text{subject to} & g(x^1,\, x^2) \leq 0, \end{array} \tag{4.1}$$

where the players' objective functions $f_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ and $f_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ and the constraint mapping $g : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^m$ are smooth. This simplified problem setting captures the most significant features of GNEP. One of those features making these problems difficult for analysis and numerical solution is their tendency to have nonisolated solutions, and this feature is specially interesting in the context of this work. GNEP setting goes back to [34]; for recent surveys, see [12, 15].

Writing down the KKT optimality systems for the two optimization problems in (4.1), and removing duplicated constraints, we obtain the following system in the primal-dual variables:

$$\begin{gathered} \frac{\partial L_1}{\partial x^1}(x^1,\, x^2,\, \mu^1) = 0, \quad \frac{\partial L_2}{\partial x^2}(x^1,\, x^2,\, \mu^2) = 0, \\ \mu^1 \geq 0, \quad \langle \mu^1,\, g(x^1,\, x^2)\rangle = 0, \quad \mu^2 \geq 0, \quad \langle \mu^2,\, g(x^1,\, x^2)\rangle = 0, \quad g(x^1,\, x^2) \leq 0, \end{gathered} \tag{4.2}$$

where $L_j : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m \to \mathbb{R}$ is the Lagrangian of the corresponding optimization problem in (4.1), i.e.,

$$L_j(x^1,\, x^2,\, \mu^j) = f_j(x^1,\, x^2) + \langle \mu^j,\, g(x^1,\, x^2)\rangle, \quad j = 1,\, 2.$$

Along with (4.1), consider its perturbed version with the canonically perturbed players' problems:

$$\begin{array}{lll} \text{minimize}_{x^1} & f_1(x^1,\, x^2) - \langle a^1,\, x^1\rangle & \qquad \text{minimize}_{x^2} & f_2(x^1,\, x^2) - \langle a^2,\, x^2\rangle \\ \text{subject to} & g(x^1,\, x^2) \leq b, & \qquad \text{subject to} & g(x^1,\, x^2) \leq b, \end{array}$$

where $a^1 \in \mathbb{R}^{n_1}$, $a^2 \in \mathbb{R}^{n_2}$ and $b \in \mathbb{R}^m$ characterize perturbations. Note that b is the same for both players, i.e., joint constraints remain remain joint after perturbation. The corresponding perturbed version of the system (4.2) has the form

$$\begin{gathered} \frac{\partial L_1}{\partial x^1}(x^1,\, x^2,\, \mu^1) = a^1, \quad \frac{\partial L_2}{\partial x^2}(x^1,\, x^2,\, \mu^2) = a^2, \\ \mu^1 \geq 0, \quad \langle \mu^1,\, g(x^1,\, x^2) - b\rangle = 0, \quad \mu^2 \geq 0, \quad \langle \mu^2,\, g(x^1,\, x^2) - b\rangle = 0, \quad g(x^1,\, x^2) \leq b. \end{gathered} \tag{4.3}$$

For a given solution $(\bar{x}^1,\, \bar{x}^2,\, \bar{\mu}^1,\, \bar{\mu}^2)$ of (4.2), define the index sets

$$A = A(\bar{x}^1,\, \bar{x}^2) = \{i = 1,\, \ldots,\, m \mid g_i(\bar{x}^1,\, \bar{x}^2) = 0\},$$

$$
\begin{gathered}
N=N(\bar{x}^1,\,\bar{x}^2)=\{1,\,\ldots,\,m\}\setminus A,\\
A_+^j=A_+^j(\bar{x}^1,\,\bar{x}^2,\,\bar{\mu}^j)=\{i\in A\mid \bar{\mu}^j>0\},\quad j=1,\,2,\\
A_0^j=A_0^j(\bar{x}^1,\,\bar{x}^2,\,\bar{\mu}^j)=A\setminus A_+^j,\quad j=1,\,2,\\
A_+=A_+(\bar{x}^1,\,\bar{x}^2,\,\bar{\mu}^1,\,\bar{\mu}^2)=A_+^1\cup A_+^2,\quad A_0=A_0(\bar{x}^1,\,\bar{x}^2,\,\bar{\mu}^1,\,\bar{\mu}^2)=A_0^1\cap A_0^2.
\end{gathered}
$$

Then near the solution in question, and for $(a^1,\,a^2,\,b)$ close enough to $(0,\,0,\,0)$, the solution set of (4.3) is the union of solution sets of the branch systems

$$
\begin{gathered}
\frac{\partial L_1}{\partial x^1}(x^1,\,x^2,\,\mu^1)=a^1,\quad \frac{\partial L_2}{\partial x^2}(x^1,\,x^2,\,\mu^2)=a^2,\quad \mu_N^1=0,\quad \mu_N^2=0,\\
\mu_{J_1}^1\geq 0,\quad \mu_{J_1}^2\geq 0,\quad g_{J_1}(x^1,\,x^2)=b_{J_1},\\
\mu_{J_2}^1=0,\quad \mu_{J_2}^2=0,\quad g_{J_2}(x^1,\,x^2)\leq b_{J_2},\\
\mu^1_{A_0^1\setminus A_0^2}\geq 0,\quad \mu^2_{A_0^2\setminus A_0^1}\geq 0,\quad g_{A_+}(x^1,\,x^2)=b_{A_+},
\end{gathered}
\tag{4.4}
$$

defined by all partitions $(J_1,\,J_2)$ of A_0.

The strict complementarity condition for GNEP KKT-type systems (4.2) consists of saying that $A_0^1=\emptyset$ and $A_0^2=\emptyset$. If this condition holds, then (4.4) reduces to the system of equations

$$
\frac{\partial L_1}{\partial x^1}(x^1,\,x^2,\,\mu^1)=a^1,\quad \frac{\partial L_2}{\partial x^2}(x^1,\,x^2,\,\mu^2)=a^2,\quad \mu_N^1=0,\quad \mu_N^2=0,\quad g_A(x^1,\,x^2)=b_A.
\tag{4.5}
$$

Assuming that the components of μ^1 and μ^2 are ordered in such a way that $\mu^1=(\mu_A^1,\,\mu_N^1)$ and $\mu^2=(\mu_A^2,\,\mu_N^2)$, we finally obtain the following system of equations characterizing x^1, x^2 and the unknown components of μ^1 and μ^2:

$$
\frac{\partial L_1}{\partial x^1}(x^1,\,x^2,\,\mu_A^1,\,0)=a^1,\quad \frac{\partial L_2}{\partial x^2}(x^1,\,x^2,\,\mu_A^2,\,0)=a^2,\quad g_A(x^1,\,x^2)=b_A
\tag{4.6}
$$

with respect to $u=(x^1,\,x^2,\,\mu_A^1,\,\mu_A^2)$. Therefore, the behavior of such solution $(\bar{x}^1,\,\bar{x}^2,\,\bar{\mu}^1,\,\bar{\mu}^2)$ is characterized by the existing results for unconstrained equations, applied to (1.7) with $\Phi:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{|A|}\times\mathbb{R}^{|A|}\to\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{|A|}$,

$$
\Phi(u)=\left(\frac{\partial L_1}{\partial x^1}(x^1,\,x^2,\,\mu_A^1,\,0),\,\frac{\partial L_2}{\partial x^2}(x^1,\,x^2,\,\mu_A^2,\,0),\,g_A(x^1,\,x^2)\right),
$$

with the right-hand side perturbation $w=(a^1,\,a^2,\,b_A)$. The basic solution of interest (of (1.7) with $w=0$) is $\bar{u}=(\bar{x}^1,\,\bar{x}^2,\,\bar{\mu}_A^1,\,\bar{\mu}_A^2)$.

Observe, however, that unlike what we had for complementarity problems in Section 3, (4.6) is a system of $n_1+n_2+|A|$ equations in $n_1+n_2+2|A|$ variables, and nonisolated solutions of this system do not need to be singular: the Jacobian

$$
\Phi'(\bar{u})=\begin{pmatrix}
\frac{\partial^2 L_1}{\partial x^1\partial x^1}(\bar{x}^1,\,\bar{x}^2,\,\bar{\mu}^1) & \frac{\partial^2 L_1}{\partial x^1\partial x^2}(\bar{x}^1,\,\bar{x}^2,\,\bar{\mu}^1) & \left(\frac{\partial g_A}{\partial x^1}(\bar{x}^1,\,\bar{x}^2)\right)^{\mathrm{T}} & 0\\
\frac{\partial^2 L_2}{\partial x^1\partial x^2}(\bar{x}^1,\,\bar{x}^2,\,\bar{\mu}^2) & \frac{\partial^2 L_2}{\partial x^2\partial x^2}(\bar{x}^1,\,\bar{x}^2,\,\bar{\mu}^2) & 0 & \left(\frac{\partial g_A}{\partial x^2}(\bar{x}^1,\,\bar{x}^2)\right)^{\mathrm{T}}\\
\frac{\partial g_A}{\partial x^1}(\bar{x}^1,\,\bar{x}^2) & \frac{\partial g_A}{\partial x^2}(\bar{x}^1,\,\bar{x}^2) & 0 & 0
\end{pmatrix}
\tag{4.7}
$$

can have full row rank. If this is the case, and if $A \neq \emptyset$, this solution is necessarily nonisolated, but at the same time, it is metrically regular. In particular the local Lipschitzian error bound holds at this solution (this result appears as the Lyusternik theorem in [19]), and it is stable subject to small enough but otherwise arbitrary right-hand side perturbations with a Lipschitzian estimate (this follows from the classical covering result for nonlinear mappings, sometimes called the Graves theorem [11, Theorem 5D.2]). These properties are readily translated to the solution $(\bar{x}^1, \bar{x}^2, \bar{\mu}^1, \bar{\mu}^2)$ of (4.2), and to the perturbed version (4.3) of the latter.

If the row rank of the Jacobian in (4.7) is not full, the system (4.5) can be studied by means of the results in [22]. Moreover, since the number of equations in this system is greater than the number of variables (unless $A = \emptyset$), the covering result from [21, Theorem 5] can be applicable, which in its turn is a corollary of the implicit function theorem obtained in [7] (see also [3]). Unlike more general theorems in [20] and [22], the result in [21, Theorem 5] establishes covering of an entire neighborhood of 0 in the space of right-hand side perturbations, and with a square-root estimate as in Corollary 2.2, provided Φ is 2-regular at $\bar{u}$ (in the sense of (1.5)) in some direction $\bar{v} \in \ker \Phi'(\bar{u})$ such that $\Phi''(\bar{u})[\bar{v}, \bar{v}] \in \operatorname{im} \Phi'(\bar{u})$.

We proceed with the case when the strict complementarity condition does not hold. For a given partition (J_1, J_2) of A_0, let the components of μ^1 and μ^2 be ordered in such a way that $\mu^1 = (\mu^1_{A_+}, \mu^1_N, \mu^1_{J_1}, \mu^1_{J_2})$, $\mu^2 = (\mu^2_{A_+}, \mu^2_N, \mu^2_{J_1}, \mu^2_{J_2})$. Introducing the slack variable $\sigma \in \mathbb{R}^{|J_2|}$, the tuples satisfying (4.4) are equivalently characterized by the system

$$\begin{gathered}
\frac{\partial L_1}{\partial x^1}(x^1, x^2, \mu^1_{A_+}, 0, \mu^1_{J_1}, 0) = a^1, \quad \frac{\partial L_2}{\partial x^2}(x^1, x^2, \mu^2_{A_+}, 0, \mu^2_{J_1}, 0) = a^2, \\
\mu^1_{J_1} \geq 0, \quad \mu^2_{J_1} \geq 0, \quad g_{J_1}(x^1, x^2) = b_{J_1}, \quad g_{J_2}(x^1, x^2) - \sigma = b_{J_2}, \quad \sigma \geq 0, \\
\mu^1_{A_0^1 \setminus A_0^2} \geq 0, \quad \mu^2_{A_0^2 \setminus A_0^1} \geq 0, \quad g_{A_+}(x^1, x^2) = b_{A_+},
\end{gathered} \tag{4.8}$$

with respect to $u = (x^1, x^2, \mu^1_{A_+}, \mu^1_{J_1}, \mu^2_{A_+}, \mu^2_{J_1}, \sigma)$. System (4.8) is a constrained equation (3.7), where $\Phi : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{|A_+|} \times \mathbb{R}^{|J_1|} \times \mathbb{R}^{|A_+|} \times \mathbb{R}^{|J_1|} \times \mathbb{R}^{|J_2|} \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{|A_+|} \times \mathbb{R}^{|J_1|} \times \mathbb{R}^{|J_2|}$,

$$\Phi(u) = \begin{pmatrix} \dfrac{\partial L_1}{\partial x^1}(x^1, x^2, \mu^1_{A_+}, 0, \mu^1_{J_1}, 0), \\ \dfrac{\partial L_2}{\partial x^2}(x^1, x^2, \mu^2_{A_+}, 0, \mu^2_{J_1}, 0), \\ g_{A_+}(x^1, x^2), \\ g_{J_1}(x^1, x^2), \\ g_{J_2}(x^1, x^2) - \sigma \end{pmatrix},$$

$$K = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times (\mathbb{R}^{|A^1_+|} \times \mathbb{R}_+^{|A_0^1 \setminus A_0^2|}) \times \mathbb{R}_+^{|J_1|} \times (\mathbb{R}^{|A^2_+|} \times \mathbb{R}_+^{|A_0^2 \setminus A_0^1|}) \times \mathbb{R}_+^{|J_1|} \times \mathbb{R}_+^{|J_2|},$$

with the right-hand side perturbation $w = (a^1, a^2, b_{J_1}, b_{A_+}, b_{J_2})$ (in the definition of K, we further assume that the components of μ^1 and μ^2 are ordered in such a way that $\mu^1_{A_+} = (\mu^1_{A^1_+}, \mu^1_{A_0^1 \setminus A_0^2})$, $\mu^2_{A_+} = (\mu^2_{A^2_+}, \mu^2_{A_0^2 \setminus A_0^1})$). The basic solution of interest (of (3.7) with $w = 0$) is $\bar{u} = (\bar{x}^1, \bar{x}^2, (\bar{\mu}^1_{A^1_+}, 0), 0, (\bar{\mu}^2_{A^2_+}, 0), 0, 0)$, and (3.7) is equivalent to (3.10).

In order to apply Corollary 2.1, consider the Jacobian

$$
\Phi'(\bar{u}) = \begin{pmatrix}
\dfrac{\partial^2 L_1}{\partial x^1 \partial x^1} & \dfrac{\partial^2 L_1}{\partial x^1 \partial x^2} & \left(\dfrac{\partial g_{A_+ \cup J_1}}{\partial x^1}\right)^{\mathrm{T}} & 0 & 0 \\
\dfrac{\partial^2 L_2}{\partial x^1 \partial x^2} & \dfrac{\partial^2 L_2}{\partial x^2 \partial x^2} & 0 & \left(\dfrac{\partial g_{A_+ \cup J_1}}{\partial x^2}\right)^{\mathrm{T}} & 0 \\
\dfrac{\partial g_{A_+}}{\partial x^1} & \dfrac{\partial g_{A_+}}{\partial x^2} & 0 & 0 & 0 \\
\dfrac{\partial g_{J_1}}{\partial x^1} & \dfrac{\partial g_{J_1}}{\partial x^2} & 0 & 0 & 0 \\
\dfrac{\partial g_{J_2}}{\partial x^1} & \dfrac{\partial g_{J_2}}{\partial x^2} & 0 & 0 & -I
\end{pmatrix},
$$

where the derivatives are computed at the same points as in (4.7) (skipped for brevity). Since $\operatorname{int} K \neq \emptyset$, condition (2.13) holds if and only if this Jacobian has full row rank, which, in its turn, is equivalent to saying that the matrix

$$
\begin{pmatrix}
\dfrac{\partial^2 L_1}{\partial x^1 \partial x^1} & \dfrac{\partial^2 L_1}{\partial x^1 \partial x^2} & \left(\dfrac{\partial g_{A_+ \cup J_1}}{\partial x^1}\right)^{\mathrm{T}} & 0 \\
\dfrac{\partial^2 L_2}{\partial x^1 \partial x^2} & \dfrac{\partial^2 L_2}{\partial x^2 \partial x^2} & 0 & \left(\dfrac{\partial g_{A_+ \cup J_1}}{\partial x^2}\right)^{\mathrm{T}} \\
\dfrac{\partial g_{A_+}}{\partial x^1} & \dfrac{\partial g_{A_+}}{\partial x^2} & 0 & 0 \\
\dfrac{\partial g_{J_1}}{\partial x^1} & \dfrac{\partial g_{J_1}}{\partial x^2} & 0 & 0
\end{pmatrix} \tag{4.9}
$$

has full row rank. By Corollary 2.1, taking into account Remark 2.1, we obtain the following

Proposition 4.1 *Let f_1, f_2 and g be twice differentiable near $(\bar{x}^1, \bar{x}^2)$, and let their second derivatives be continuous at $(\bar{x}^1, \bar{x}^2)$. Let $(\bar{\mu}^1, \bar{\mu}^2)$ be such that $\bar{z} = (\bar{x}^1, \bar{x}^2, \bar{\mu}^1, \bar{\mu}^2)$ is a solution of* (4.2)*. Let there exist a partition (J_1, J_2) of A_0 such that the matrix in* (4.9) *has full row rank.*

Then there exist a cone $C \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m$ and $\theta > 0$ such that $\operatorname{int} C \neq \emptyset$, and for every $y = (a^1, a^2, b) \in C$ close enough to $(0, 0, 0)$, there exists a solution $z(y) = (x^1(y), x^2(y), \mu^1(y), \mu^2(y))$ of (3.2) *satisfying $\|z(y) - \bar{z}\| \leq \|y\|/\theta$.*

Taking $J_1 = \emptyset$, $J_2 = A_0$, reduces the matrix in (4.9) to

$$
\begin{pmatrix}
\dfrac{\partial^2 L_1}{\partial x^1 \partial x^1} & \dfrac{\partial^2 L_1}{\partial x^1 \partial x^2} & \left(\dfrac{\partial g_{A_+}}{\partial x^1}\right)^{\mathrm{T}} & 0 \\
\dfrac{\partial^2 L_2}{\partial x^1 \partial x^2} & \dfrac{\partial^2 L_2}{\partial x^2 \partial x^2} & 0 & \left(\dfrac{\partial g_{A_+}}{\partial x^2}\right)^{\mathrm{T}} \\
\dfrac{\partial g_{A_+}}{\partial x^1} & \dfrac{\partial g_{A_+}}{\partial x^2} & 0 & 0
\end{pmatrix}.
$$

If this matrix has full row rank, Proposition 4.1 is automatically applicable with the specified partitions. Observe that if $A_0 = \emptyset$, which is still weaker than the strict complementarity condition, this matrix takes the form

$$\left(\begin{array}{cccc} \dfrac{\partial^2 L_1}{\partial x^1 \partial x^1} & \dfrac{\partial^2 L_1}{\partial x^1 \partial x^2} & \left(\dfrac{\partial g_A}{\partial x^1}\right)^{\mathrm{T}} & 0 \\ \dfrac{\partial^2 L_2}{\partial x^1 \partial x^2} & \dfrac{\partial^2 L_2}{\partial x^2 \partial x^2} & 0 & \left(\dfrac{\partial g_A}{\partial x^2}\right)^{\mathrm{T}} \\ \dfrac{\partial g_A}{\partial x^1} & \dfrac{\partial g_A}{\partial x^2} & 0 & 0 \end{array}\right),$$

the matrix appearing in [25, Proposition 2] as a part of the condition ensuring the local Lipschitzian error bound for (4.2).

Observe that here, not only the assumptions of Corollary 2.1 but even the stronger Robinson's condition can be satisfied for some branches, being equivalent to saying that the Mangasarian–Fromovitz constraint qualification (MFCQ) holds for the system (4.8) at $\bar{u}$, and with $(a^1, a^2, b) = (0, 0, 0)$. The following is the model example widely used in GNEP literature.

Example 4.1 ([12, Example 1.1]) Consider the GNEP (4.1) with $f_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f_1(x^1, x^2) = (x^1 - 1)^2$, $f_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f_2(x^1, x^2) = (x^1 - 1/2)^2$, $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g(x^1, x^2) = x_1 + x_2 - 1$. The solution set of the related KKT-type system (4.2) has the form

$$\left\{ (x^1, x^2, \mu^1, \mu^2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \;\middle|\; \begin{array}{c} x^1 = t,\ x^2 = 1 - t,\ \mu^1 = 2(1 - t),\ \mu^2 = 2(t - 1/2), \\ t \in [1/2, 1] \end{array} \right\}.$$

For $y = (a^1, a^2, b) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ close enough to $(0, 0)$, the perturbed KKT-type system (4.3) has the form

$$\begin{gathered} 2(x^1 - 1) - a^1 + \mu^1 = 0, \quad 2(x^2 - 1/2) - a^2 + \mu^2 = 0, \\ \mu^1 \geq 0, \quad \mu^1(x^1 + x^2 - 1 - b) = 0, \quad \mu^2 \geq 0, \quad \mu^2(x^1 + x^2 - 1 - b) = 0, \quad x^1 + x^2 \leq 1 + b. \end{gathered}$$

The solution set of this system is

$$\left\{ (x^1, x^2, \mu^1, \mu^2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \;\middle|\; \begin{array}{c} x^1 = t,\ x^2 = 1 + b - t, \\ \mu^1 = 2(1 - t) + a^1,\ \mu^2 = 2\left(t - \dfrac{1}{2} - b\right) + a^2, \\ t \in \left[\dfrac{1}{2} - \dfrac{1}{2}a^2 + b,\ 1 + \dfrac{1}{2}a^1\right] \end{array} \right\}.$$

Hence, all solutions of the unperturbed KKT-type system are stable subject to arbitrary perturbations of the specified kind if they are small enough.

At any solution $((t, 1 - t), (2(1 - t), 2(t - 1/2)))$ with $t \in (1/2, 1)$, satisfying the strict complementarity condition, the Jacobian in (4.7) appears to be

$$\Phi'(\bar{u}) = \left(\begin{array}{cccc} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{array}\right).$$

It has full row rank, and hence, stability of this solution subject to arbitrary perturbations of the type under consideration follows from the Banach open mapping theorem, since Φ is affine. If we add some higher-order terms in (4.1), vanishing at $(t,\, 1-t)$, this conclusion will remain valid for small perturbations, but with the reference to the Graves theorem.

Consider now the solution $((1/2,\, 1/2),\, (1,\, 0))$ violating strict complementarity. Then $N = \emptyset$, $A_+^2 = \emptyset$, $A_1^0 = \emptyset$, implying that $A_0 = \emptyset$, and hence, the system (3.6) reduces to

$$2(x^1 - 1) + \mu^1 = a_1, \quad 2(x^2 - 1/2) + \mu^2 = a^2, \quad x^1 + x^2 = 1 + b, \quad \mu^2 \geq 0,$$

giving a single branch of the solution set. Being considered as a constraint system, and with $(a^1,\, a^2,\, b) = (0,\, 0,\, 0)$, this system satisfies MFCQ at the solution in question, which is equivalent to saying that Robinson's condition holds for the corresponding constrained equation. Therefore, even after adding higher-order terms in (4.1), vanishing at $(1/2,\, 1/2)$, the solution in question will be stable subject to arbitrary small perturbations of the specified kind, according to Robinson's stability theorem.

Solution $((1,\, 0),\, (0,\, 1))$ can be considered similarly, and with similar conclusions.

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