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A Metric for Total Tardiness Minimization

A. A. Lazarev[*,**,***,****,a], **P. S. Korenev**[*,**,b], **and A. A. Sologub**[*,**,c]

[*] *Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia*
[**] *Lomonosov State University, Moscow, Russia*
[***] *Moscow Institute of Physics and Technology, Moscow, Russia*
[****] *Higher School of Economics (National Research University), Moscow, Russia*
e-mail: [a]*jobmath@mail.ru,* [b]*pkorenev@rambler.ru,* [c]*sologub10@gmail.com*

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Abstract—In this paper we consider the NP-hard $1|r_j|\sum T_j$ scheduling problem, suggesting a polynomial algorithm to find its approximate solution with the guaranteed absolute error. The algorithm employs a metric introduced in the parameter space. In addition, we study the possible application of such an approach to other scheduling problems.

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1. INTRODUCTION

Consider a set $N = \{1, 2, \ldots, n\}$ of n requests that have to be served by a single machine. The machine is ready to start servicing at the time $t_0 = 0$ and may simultaneously serve at most one request. Any interruptions during servicing are forbidden. For each request $j \in N$, the given parameters are the arrival time r_j, the service time p_j and the directive deadline d_j. A schedule $\pi = \{j_1, j_2, \ldots, j_n\}$ defines an order to serve the requests. A natural approach is to study early schedules that satisfy

$$C_{j_1}(\pi) = r_{j_1} + p_{j_1},$$
$$C_{j_k}(\pi) = \max\{r_{j_k}, C_{j_{k-1}}(\pi)\} + p_{j_k}, \quad k = 2, 3, \ldots, n,$$

where $C_j(\pi)$ denotes the service termination time of request j in a schedule π. It is required to construct an optimal schedule π^* that minimizes the objective function in the form of the total tardiness $\sum_{j\in N} T_j(\pi)$, where $T_j(\pi) = \max\{0, C_j(\pi) - d_j\}$ means the tardiness of request j in a schedule π. Subsequently, we will omit the dependence on π whenever no confusion occurs. This problem is NP-hard [5] and denoted by $1|r_j|\sum T_j$ [6].

In fact, the problem $1|r_j|\sum T_j$ is completely described by $3n$ parameters, namely, the directive deadlines, the service times and the arrival times of n requests. Throughout the paper, we will consider example A of this problem with the given $3n$ parameters $\left\{r_j^A, p_j^A, d_j^A, j = 1, 2, \ldots, n\right\}$, which completely characterize the problem.

In the special case $r_j = 0$, $j \in N$, the total tardiness problem was earlier solved in [8] using the polynomial approximate algorithm with a complexity of $O(\frac{n^7}{\varepsilon})$ operations. For this case another well-known solution method is the pseudopolynomial algorithm that has complexity $O\left(n^4 \sum p_j\right)$, see [7]. If

$$p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n,$$
$$d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n,$$

then the complexity of the pseudopolynomial algorithm can be reduced to $O\left(n^2 \sum p_j\right)$ operations [9]. For the case $1|r_j, p_j = p|\sum T_j$, we also mention the polynomial algorithm with $O(n^7)$ operations that was suggested by P. Baptiste in [4].

The present paper introduces an approximate solution approach to the problem $1|r_j|\sum T_j$ with the guaranteed error that employs a metric in the parameter space. We also analyze the possible application of this approach to other scheduling problems. Finally, we test the suggested approach using a series of numerical experiments.

2. METRIC FOR PARAMETER SPACE

The problem $1|r_j|\sum T_j$ is completely described by the $3n$ parameters above. And so, we may consider the examples of this problem as the points in the $3n$-dimensional parameter space $\Omega = \{r_1, \ldots, r_n, p_1, \ldots, p_n, d_1, \ldots, d_n\}$.

Lemma 1. *Let examples A and B be characterized by the same service times and the same directive deadlines, i.e.,*

$$p_j^A = p_j^B, \quad d_j^A = d_j^B, \quad j \in N.$$

Then for any schedule π we have

$$\left| \sum_{j \in N} T_j^A(\pi) - \sum_{j \in N} T_j^B(\pi) \right| \leqslant n \max_{j \in N} \left| r_j^A - r_j^B \right|. \tag{1}$$

Proof of Lemma 1. The definition of tardiness and the well-known inequality

$$|\max\{a, b\} - \max\{c, d\}| \leqslant \max\{|a - c|, |b - d|\}, \quad \forall a, b, c, d \in \mathbb{R}, \tag{2}$$

yield

$$\begin{aligned} \left| \sum_{j \in N} T_j^A - \sum_{j \in N} T_j^B \right| &\leqslant \sum_{j \in N} \left| C_j^A - C_j^B + d_j^B - d_j^A \right| \\ &\leqslant \sum_{j \in N} \left| C_j^A - C_j^B \right| + \sum_{j \in N} \left| d_j^A - d_j^B \right|. \end{aligned} \tag{3}$$

Owing to the same directive deadlines,

$$\left| \sum_{j \in N} T_j^A - \sum_{j \in N} T_j^B \right| \leqslant \sum_{j \in N} \left| C_j^A - C_j^B \right|. \tag{4}$$

Taking into account the properties of the early schedules, note that

$$\left| C_{j_1}^A - C_{j_1}^B \right| = \left| r_{j_1}^A - r_{j_1}^B \right| \leqslant \max_{j \in N} \left| r_j^A - r_j^B \right|,$$

$$\left| C_{j_k}^A - C_{j_k}^B \right| \leqslant \max\left\{ \left| r_{j_k}^A - r_{j_k}^B \right|, \left| C_{j_{k-1}}^A - C_{j_{k-1}}^B \right| \right\} \leqslant \max_{j \in N} \left| r_j^A - r_j^B \right|, \quad k = 2, \ldots, n.$$

These conditions jointly with inequality (4) give the desired result.

Lemma 2. *Let examples A and B be characterized by the same arrival times and the same directive deadlines, i.e.,*

$$r_j^A = r_j^B, \quad d_j^A = d_j^B, \quad j \in N.$$

Then for any schedule π we have

$$\left|\sum_{j\in N} T_j^A(\pi) - \sum_{j\in N} T_j^B(\pi)\right| \leqslant n \sum_{j\in N} \left|p_j^A - p_j^B\right|. \tag{5}$$

Proof of Lemma 2. Inequality (4) also holds under the hypotheses of the lemma, that is,

$$\left|\sum_{j\in N} T_j^A - \sum_{j\in N} T_j^B\right| \leqslant \sum_{j\in N} \left|C_j^A - C_j^B\right|.$$

Using the properties of the early schedules and the same arrival times, we obtain

$$\left|C_{j_1}^A - C_{j_1}^B\right| = \left|p_{j_1}^A - p_{j_1}^B\right| \leqslant \sum_{j\in N} \left|p_j^A - p_j^B\right|,$$
$$\left|C_{j_k}^A - C_{j_k}^B\right| \leqslant \left|p_{j_k}^A - p_{j_k}^B\right| + \left|C_{j_{k-1}}^A - C_{j_{k-1}}^B\right| \leqslant \sum_{j\in N} \left|p_j^A - p_j^B\right|, \quad k = 2, \ldots, n.$$

These conditions jointly with inequality (4) give the desired result.

Lemma 3. *Let examples A and B be characterized by the same arrival times and the same service times, i.e.,*

$$r_j^A = r_j^B, \quad p_j^A = p_j^B, \quad j \in N.$$

Then for any schedule π we have

$$\left|\sum_{j\in N} T_j^A(\pi) - \sum_{j\in N} T_j^B(\pi)\right| \leqslant \sum_{j\in N} \left|d_j^A - d_j^B\right|. \tag{6}$$

Proof of Lemma 3. Under the hypotheses of this lemma, $C_{j_k}^A = C_{j_k}^B$, $k \in N$, and inequality (3) acquires the form

$$\left|\sum_{j\in N} T_j^A - \sum_{j\in N} T_j^B\right| \leqslant \sum_{j\in N} \left|d_j^A - d_j^B\right|.$$

In other words, the statement of Lemma 3 is true.

Theorem 1. *The function*

$$\rho(A,B) = n \max_{j\in N} \left|r_j^A - r_j^B\right| + n \sum_{j\in N} \left|p_j^A - p_j^B\right| + \sum_{j\in N} \left|d_j^A - d_j^B\right| \tag{7}$$

that is defined on the example space $\Omega \times \Omega$ satisfies the axioms of a metric.

Proof of Theorem 1. Obviously, the function $\rho(A,B)$ is symmetric and nonnegative; moreover, $\rho(A,B) = 0$ if and only if $A = B$. And the triangle inequality follows immediately from the properties of the absolute value of the sum of two numbers.

Lemma 4. *For any examples A and B and any schedule π, we have the inequality*

$$\left|\sum_{j\in N} T_j^A - \sum_{j\in N} T_j^B\right| \leqslant \rho(A,B). \tag{8}$$

Proof of Lemma 4. Let example C have the same arrival times and the same service times as example A and the same directive deadlines as example B. Next, let example D have the same arrival times as example A and the same directive deadlines and the same service times as example B. It appears from Lemmas (1)–(3) that

$$\left|\sum_{j\in N} T_j^A - \sum_{j\in N} T_j^B\right| \leqslant \left|\sum_{j\in N} T_j^B - \sum_{j\in N} T_j^D\right| + \left|\sum_{j\in N} T_j^D - \sum_{j\in N} T_j^C\right|$$
$$+\left|\sum_{j\in N} T_j^A - \sum_{j\in N} T_j^C\right| \leqslant n\max_{j\in N}\left|r_j^A - r_j^B\right| + n\sum_{j\in N}\left|p_j^A - p_j^B\right| + \sum_{j\in N}\left|d_j^A - d_j^B\right| = \rho(A,B).$$

3. PARAMETER TRANSFORMATION METHOD

Theorem 2. *Let π^A and π^B be the optimal schedules for examples A and B. Then*

$$\sum_{j\in N} T_j^A(\pi^B) - \sum_{j\in N} T_j^A(\pi^A) \leqslant 2\rho(A,B). \tag{9}$$

Proof of Theorem 2. Using Lemma 4 we obtain

$$\sum_{j\in N} T_j^A(\pi^B) - \sum_{j\in N} T_j^A(\pi^A)$$
$$= \left(\sum_{j\in N} T_j^A(\pi^B) - \sum_{j\in N} T_j^B(\pi^B)\right) + \left(\sum_{j\in N} T_j^B(\pi^B) - \sum_{j\in N} T_j^B(\pi^A)\right)$$
$$+\left(\sum_{j\in N} T_j^B(\pi^A) - \sum_{j\in N} T_j^A(\pi^A)\right) \leqslant 2\rho(A,B).$$

This theorem allows solving the problem $1|r_j|\sum T_j$ by a procedure called the parameter transformation method. Its main idea is to employ the optimal schedule of some (pseudo)polynomial example B as the schedule for example A. Owing to Theorem 2 the error of this solution can be estimated using the function $\rho(A,B)$. It seems natural to construct example B by minimizing the function $\rho(A,B)$. Therefore, the problem $1|r_j|\sum T_j$ is replaced with the metric minimization problem.

Consider the case where example B must belong to a certain polynomially or pseudopolynomially solvable class of examples that is defined by the system of inequalities

$$\mathcal{A}\times R^B + \mathcal{B}\times P^B + \mathcal{C}\times D^B \leqslant H.$$

Here $R^B = (r_1^B,\dots,r_n^B)^\top$, $P^B = (p_1^B,\dots,p_n^B)^\top$, $D^B = (d_1^B,\dots,d_n^B)^\top$, with $p_j^B \geqslant 0$ and $r_j^B \geqslant 0$, $j\in N$; $^\top$ denotes transposition; $\mathcal{A},\mathcal{B}$, and $\mathcal{C}$ are matrices of dimensions $m\times n$, while H is a column vector of m elements.

In this case the metric minimization problem can be rewritten as the linear programming problem

$$\min n\times(y^r - x^r) + n\times\sum_{j\in N}\left(y_j^p - x_j^p\right) + \sum_{j\in N}\left(y_j^d - x_j^d\right), \tag{10}$$

subject to the constraints

$$
\begin{gathered}
x^r \leqslant r_j^A - r_j^B \leqslant y^r, \quad j \in N, \\
x_j^p \leqslant p_j^A - p_j^B \leqslant y_j^p, \quad j \in N, \\
x_j^d \leqslant d_j^A - d_j^B \leqslant y_j^d, \quad j \in N, \\
0 \leqslant r_j^B, \quad 0 \leqslant p_j^B, \quad j \in N, \\
\mathcal{A} \times R^B + \mathcal{B} \times P^B + \mathcal{C} \times D^B \leqslant H.
\end{gathered}
$$

This problem contains $(7n+2)$ unknown variables, namely, r_j^B, p_j^B, d_j^B, x_j^p, y_j^p, x_j^d, y_j^d, x^r, and y^r, where $j \in N$.

Nevertheless, in many situations the function $\rho(A, B)$ is separable and there is no need for using linear programming methods, which appreciably simplifies minimization.

3.1. Application of the Parameter Transformation Method to Other Scheduling Problems

The described method is not rigidly bound to the form of the objective function. In other words, this method can be adopted to solve other scheduling problems. We extend Theorem 2 to the case of a general-form objective function $F(\pi)$ as follows.

Theorem 3. *Let $F(\pi)$ be an arbitrary objective function and let $\rho(A, B)$ be a metric function that satisfies the inequality*

$$\left| F^A(\pi) - F^B(\pi) \right| \leqslant \rho(A, B) \tag{11}$$

for any A, B, and π. In addition, let π^A and π^B be the optimal schedules for examples A and B, respectively. Then

$$F^A\left(\pi^B\right) - F^A\left(\pi^A\right) \leqslant 2\rho(A, B). \tag{12}$$

Proof of Theorem 3. Is the same as the proof of Theorem 2 with $\sum_{j \in N} T_j$ replaced by F.

Consequently, to use the parameter transformation method it suffices to construct the function $\rho(A, B)$ satisfying inequality (11). Such functions were constructed earlier for the problems $1||\sum T_j$ and $1|r_j|L_{\max}$ in the papers [1, 3], respectively. Below we will suggest possible forms of these functions for the general cases of additive and maximal objective functions.

Lemma 5. *For the additive objective function*

$$F(\pi) = \sum_{j \in N} \phi_j(\pi, r_1, \ldots, r_n, p_1, \ldots, p_n, d_j), \tag{13}$$

the function

$$\rho(A, B) = \sum_{j \in N} \sum_{i \in N} \left(R_{ji} \left| r_j^A - r_j^B \right| + P_{ji} \left| p_j^A - p_j^B \right| \right) + \sum_{j \in N} D_j \left| d_j^A - d_j^B \right| \tag{14}$$

satisfies inequality (11). Here R_{ji} and P_{ji} are the Lipschitz constants of the function ϕ_i in the variables r_j and p_j, respectively, while D_j is the Lipschitz constant of the function ϕ_j in the variable $d_j (i, j \in N)$.

Lemma 6. *For the maximal objective function*

$$F(\pi) = \max_{j \in N} \phi_j(\pi, r_1, \dots, r_n, p_1, \dots, p_n, d_j), \tag{15}$$

the function

$$\rho(A, B) = \sum_{j \in N} \left(R_j \left| r_j^A - r_j^B \right| + P_j \left| p_j^A - p_j^B \right| \right) + D \max_{j \in N} \left| d_j^A - d_j^B \right| \tag{16}$$

satisfies inequality (11)*. Here R_j and P_j are the largest Lipschitz constants of the functions ϕ_i in the variables r_j and p_j, respectively, while D is the largest Lipschitz constant of the functions ϕ_j in the variable $d_j (i, j \in N)$.*

Note that functions (14) and (16) are separable, which appreciably simplifies their minimization.

4. NUMERICAL EXPERIMENTS

A series of numerical experiments was performed to estimate the efficiency of the suggested scheme. The search classes of the polynomially solvable examples are presented by Table 1.

For the first three classes the solution is the schedule with the nondescending order of the free parameter values. For the two last classes the solution algorithms were described in [2, 4]; their complexities make up $O(n^7)$ and $O(n^4 \sum p_j)$ operations, respectively.

To find the polynomially solvable example B within the above classes that is closest to a given example, we have to minimize the functions

$$f(r) = n \times \max_{j \in N} \left| r_j^A - r \right|; \tag{17}$$

$$g(p) = n \times \sum_{j=1}^{n} \left| p_j^A - p \right|; \tag{18}$$

$$h(d) = \sum_{j \in N} \left| d_j^A - d \right|. \tag{19}$$

Lemma 7. 1) *The minimum of function* (17) *is achieved at the point $r = \frac{r_{\max}^A + r_{\min}^A}{2}$, where $r_{\max}^A = \max_{j \in N} r_j^A$ and $r_{\min}^A = \min_{j \in N} r_j^A$.*

2) *The minimum of function* (18) *is achieved at the point $p \in \left\{ p_1^A, \dots, p_n^A \right\}$.*

3) *The minimum of function* (19) *is achieved at the point $d \in \left\{ d_1^A, \dots, d_n^A \right\}$.*

Table 1. The classes of examples used in numerical experiments

Class of examples	Metric between example B from the class and an arbitrary example A
$\{\mathcal{PR} : p_j = p, r_j = r, j \in N\}$	$\rho(A, B) = n \times \sum_{j=1}^{n} \left| p_j^A - p \right| + n \times \max_{j \in N} \left| r_j^A - r \right|$
$\{\mathcal{PD} : p_j = p, d_j = d, j \in N\}$	$\rho(A, B) = n \times \sum_{j \in N} \left| p_j^A - p \right| + \sum_{j \in N} \left| d_j^A - d \right|$
$\{\mathcal{RD} : r_j = r, d_j = d, j \in N\}$	$\rho(A, B) = n \times \max_{j \in N} \left| r_j^A - r \right| + \sum_{j \in N} \left| d_j^A - d \right|$
$\{\mathcal{P} : p_j = p, j \in N\}$	$\rho(A, B) = n \times \sum_{j \in N} \left| p_j^A - p \right|$
$\{\mathcal{R}0 : r_j = 0, j \in N\}$	$\rho(A, B) = n \times \max_{j \in N} \left| r_j^A - r \right|$

Proof of Lemma 7. The function $f(r)$ can be written as

$$n \times \max_{j \in N} \left| r_j^A - r \right| = n \max \left\{ r - r_{\min}^A, r_{\max}^A - r \right\}$$
$$= n \left(\frac{r_{\max}^A - r_{\min}^A}{2} + \left| r - \frac{r_{\max}^A + r_{\min}^A}{2} \right| \right).$$

Obviously, it has the minimum at the point $\frac{r_{\max} + r_{\min}}{2}$.

Let the function $g(p)$ be minimized at the point p_0. In this case, either $g'(p_0) = 0$ or $p_0 \in \left\{ p_1^A, \ldots, p_n^A \right\}$. Recall that $g(p)$ represents a piecewise linear function; hence, its vanishing derivative implies that the function is constant on some interval $\left[p_k^A, p_{k+1}^A \right]$, $k = 1, \ldots, n-1$. And the boundary points p_k^A and p_{k+1}^A are also the minimum points.

The last statement of Lemma 7 regarding the minimum of the function $h(d)$ can be established by analogy.

Note that several series of numerical experiments were performed. All series involved the examples with the uniformly distributed parameters on the intervals $[1, 100]$ for p_j^A, $\left[p_j, \sum_{j \in N} p_j \right]$ for d_j^A, and $[0, d_j - p_j]$ for r_j^A.

The first series of the experiments was intended to estimate the difference between the right- and left-hand sides of the inequality from Lemma 4. This difference characterizes the error of the method. For each $n = 10, 20, \ldots, 100$, actually 10 000 pairs of examples were generated. In these experiments the schedules were generated randomly. The quantity $\frac{\left| \sum_{j \in N} T_j^A - \sum_{j \in N} T_j^B \right|}{\rho(A,B)}$ was calculated for each pair. In addition, the percentage contributions of the metric terms that depend on the service times, directive deadlines and arrival times were calculated in order to identify the parameters having the major effect on the metric function.

Table 2 shows the results. The mean value of $\frac{\left| \sum_{j \in N} T_j^A - \sum_{j \in N} T_j^B \right|}{\rho(A,B)}$ varies by 5–10% under increasing n, while the metric terms that depend on the service times, directive deadlines and arrival times make approximate contributions of 35%, 20%, and 45%, respectively, to the metric function.

The second series of experiments was intended to test the parameter transformation method. The experiments were organized as follows. For each $n = 4, 5, \ldots, 10$, actually 10 000 examples were generated. The above-mentioned scheme was applied to each example to construct the approximate solution with the objective function F_e, and then the exact solution with the objective function F^* was obtained using the branch-and-bound algorithm. Next, the absolute error $\delta = F_e - F^*$ of the scheme was compared with its upper estimate (9) by calculating the ratio

$$\Delta = \frac{F_e - F^*}{2\rho(A, B)}. \tag{20}$$

The numerical experiments yielded the following outcomes. For the polynomially solvable examples searched within the class RD, the mean error of solution grows from 20% to 30% with respect to the upper estimate (9) under increasing n. Therefore, the schedule with the ascending order of the service times mismatches the examples with the given parameter distribution. For the other classes, the mean error is independent of n, making up few percent of the maximum theoretical error. This really small error occurred owing to the exact solution of the problem by the parameter transformation method almost in 20% cases. The relationship between the mean error Δ and n can be observed in Table 3.

Table 2. The mean difference between the objective functions and the percentage contributions of the metric terms

n	$\dfrac{\lvert\sum T_j^A - T_j^B\rvert}{\rho}$	$\dfrac{\rho_r}{\rho}$	$\dfrac{\rho_p}{\rho}$	$\dfrac{\rho_d}{\rho}$
10	11.7%	35.6%	42.3%	20.6%
20	10.4%	39.7%	39.4%	19.4%
40	8.9%	42.4%	37.4%	18.6%
60	7.8%	43.6%	36.6%	18.3%
80	7.3%	44.4%	34.4%	18.0%
100	6.7%	44.9%	35.7%	17.9%

Table 3. The mean experimental error stated as a percent of the theoretical error

n	$\mathcal{PR}$	$\mathcal{PD}$	$\mathcal{RD}$	$\mathcal{P}$	$\mathcal{R}0$
4	2.5%	4.6%	20.8%	1.8%	2.9%
5	2.6%	4.8%	23.1%	1.9%	2.8%
6	2.6%	4.6%	24.6%	1.9%	2.7%
7	2.6%	4.7%	26.0%	1.9%	2.5%
8	2.5%	4.6%	27.0%	2.0%	2.3%
9	2.4%	4.7%	27.9%	2.0%	2.2%
10	2.4%	4.6%	28.6%	1.9%	2.1%

5. CONCLUSIONS

This paper has presented a new approximate solution method for the total tardiness problem. The whole idea of the approach consists in introducing a metric in the parameter space and using an auxiliary closest example for a given example in terms of this metric.

Among possible directions of further research, we mention the development of more efficient metrics for scheduling problems and the search for new polynomially or pseudopolynomially solvable classes of examples that are applicable with the parameter transformation method.

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